Stochastic Completeness of Graphs

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Abstract

We analyze the stochastic completeness of a heat kernel on graphs which is a function of three variables: a pair of vertices and a continuous time, for infinite, locally finite, connected graphs. For general graphs, a sufficient condition for stochastic completeness is given in terms of the maximum valence on spheres about a fixed vertex. That this result is optimal is shown by studying a particular family of trees. We also prove a lower bound on the bottom of the spectrum for the discrete Laplacian and use this lower bound to show that in certain cases the Laplacian has empty essential spectrum.

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Chapter 1

Introduction

1.1 Introduction and Statement of Results

The purpose of this thesis is to model a diffusion process on infinite graphs which is analogous to the flow of heat on an open Riemannian manifold. In particular, we are interested in the stochastic completeness of this process and a precise borderline for when the stochastic completeness breaks down. Stochastic completeness can be formulated in several equivalent ways: as a property of the heat kernel, as the uniqueness of bounded solutions for the heat equation, or as the non-existence of bounded, non-zero, λ-harmonic (or λ-subharmonic) functions for a negative constant λ. In studying this property, we have benefited tremendously from the survey article of Grigor'yan [9] which discusses, in great depth, stochastic completeness in the case of Riemannian manifolds. For graphs, the starting point for our work is the paper of Dodziuk and Mathai [6] where it is shown that any graph whose valence is uniformly bounded above by a constant is stochastically complete.

In the first part, we give a construction of the heat kernel on a general graph via an exhaustion argument. This is analogous to the construction on

open Riemannian manifolds and we follow the presentation given in [3]. We also point out that one can construct the heat kernel by utilizing the spectral theorem but the two constructions result in the same kernel [3]. Next, we introduce the notion of stochastic completeness and prove the equivalence of the various formulations mentioned above. This material is adapted from [9]. We then turn our focus to a class of trees which we call *model* because their definition is analogous to that of rotationally symmetric or model manifolds. The defining property of these trees is that they contain a vertex x_0, which we call the *root* for the model, such that the valence at every other vertex depends only on the distance from x_0. Let $m(r)$ denote this common number where r denotes the distance from x_0. The main result of this section says that such trees will be stochastically complete if and only if $\sum_{r=0}^{\infty} \frac{1}{m(r)} = \infty$. We note here the similarity between this criterion and the one for the recurrence of the Brownian motion on a complete, model surface [11, 9].

We then consider general trees and prove that if a tree contains a stochastically incomplete model subtree then it must be stochastically incomplete. We first prove this in the special case when the branching of the general tree is growing rapidly in all directions from the root, then in the general case where the branching is growing rapidly in just some direction. Also, we show that if a tree is contained in a stochastically complete model tree then it must be stochastically complete. The proof of this fact follows from the more general statement that, for any graph G, if there exists a vertex x_0 such that the maximum valence of vertices on spheres centered at x_0 is not growing too rapidly then G must be stochastically complete.

Next, we prove theorems analogous to a result of Cheeger and Yau [1] which compare the heat kernel on a model tree to the heat kernel on a general tree. Let T_n be a model tree with root vertex x_0 where $n(r)$ is $m(r) - 1$, that is, one less than the common valence of vertices on the sphere of radius r about x_0. We denote the heat kernel on T_n by $\rho_t(x_0, x)$ and first show that, as a

function of x, $\rho_t(x_0, x)$ only depends on the distance from x_0. That is, if we let $r(x) = d(x, x_0)$, where $d(x, x_0)$ denotes the distance between x and x_0, then we can write $\rho_t(r) = \rho_t(x_0, r(x))$. Let T denote a general tree with heat kernel $p_t(x_0, x)$. Then, if the branching on T is growing faster in all directions from x_0 then the branching on T_n, we show that $p_t(x_0, x) \leq \rho_t(r(x))$. In a similar fashion, if $T \subseteq T_n$, then $\rho_t(r(x)) \leq p_t(x_0, x)$.

We finish this chapter by considering an operator related to the combinatorial Laplacian that we study throughout the rest of the thesis. This operator, referred to here as the *bounded Laplacian*, arises when one assigns the standard weight to the edges of a graph. We show here that the heat kernel associated to this Laplacian is stochastically complete for every graph G. In particular, bounded solutions for the combinatorial heat equation involving the bounded Laplacian are unique.

In the final part of the thesis we study the spectrum of the Laplacian on a general graph. Specifically, we introduce $\lambda_0(\Delta)$, the bottom of the spectrum of the Laplacian, and prove a characterization of it in terms of the existence of positive λ-harmonic functions. That is, there always exist positive functions satisfying $\Delta u = \lambda u$ for $\lambda \leq \lambda_0(\Delta)$ whereas such functions never exist for $\lambda > \lambda_0(\Delta)$ [14, 4]. We then prove a lower bound on $\lambda_0(\Delta)$ under a geometric assumption on G. Specifically, we assume that if we fix a vertex x_0, then at every other vertex x of G the ratio of the difference of the number of edges leaving x and going away from x_0 and the number of edges going towards x_0 divided by the total valence at x is bounded below by a positive constant. The lower bound is then given in terms of this constant. In the final section, we use this lower bound to prove that, with the additional assumption that the minimum valence on spheres about x_0 is going to infinity as one moves away from the fixed vertex, the Laplacian on the graph has empty essential spectrum. This is analogous to the result of Donnelly and Li for complete, simply connected, negatively curved Riemannian manifolds [7].

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1.2 Notation and Fundamentals

In this section we fix our notation and state and prove some basic lemmas which will be used throughout. In general, $G = (V, E)$ will denote an infinite, locally finite, connected graph where $V = V(G)$ is the set of vertices of G and $E = E(G)$ the set of edges. At times, we abuse notation and write $x \in G$ when x is a vertex of G. We will use the notation $x \sim y$ to indicate that an edge connects the vertices x and y while $[x, y]$ will denote the *oriented* edge from x to y. In general, to be able to write down certain formulas, we will assume that our graphs come with an orientation, that is, that every edge is oriented, but none of our results depend on the choice of this orientation. We use the notation $m(x)$ to indicate the *valence* at a vertex x, that is, the number of edges emanating from x.

For a finite subgraph D of G, we let $\mathrm{Vol}(D)$ denote the *volume* of D which we take, by definition, to be the number of vertices of D. That is,

$$\mathrm{Vol}(D) = \#\{x \mid x \in V(D)\}.$$

We also use the usual notion of distance between two vertices of the graph. Specifically, $d(x, y)$ will denote the number of edges in the shortest path connecting the vertices x and y.

We call f a *function on the graph G* if it is a mapping $f : V \to \mathbf{R}$. The set of all such functions will be denoted by $C(V)$. We will also use the notation

$C_0(V)$ for the space of all finitely supported functions on G and $\ell^2(V)$ for the space of all square summable functions. That is, $\ell^2(V)$ consists of all functions on G which satisfy

$$\sum_{x \in V} f(x)^2 < \infty$$

and is a Hilbert space with inner product

$$\langle f, g \rangle = \sum_{x \in V} f(x)g(x).$$

Similarly, we let $\ell^2(\tilde{E})$ denote the Hilbert space of all square summable functions on oriented edges satisfying the relation $\varphi([x,y]) = -\varphi([y,x])$ with inner product

$$\langle \varphi, \psi \rangle = \sum_{[x,y] \in \tilde{E}} \varphi([x,y])\psi([x,y])$$

where $\tilde{E}$ denotes the set of all oriented edges of G.

We now recall the definitions of the *coboundary* and *Laplacian* operators and state and prove an analogue of Green's Theorem for them. The coboundary operator d takes a function on the vertices of G and sends it to a function on the oriented edges of G defined by:

$$df([x,y]) = f(y) - f(x).$$

The combinatorial Laplacian Δ operates on functions on G by the formula:

$$\Delta f(x) = \sum_{y \sim x} \big(f(x) - f(y)\big) = m(x)f(x) - \sum_{y \sim x} f(y) \tag{1.1}$$

where the summation is taken over all vertices y such that $y \sim x$ forms an edge in G. If the Laplacian is applied to a function of more than one variable then we will put the variable in which it is applied as a subscript when necessary. For a constant λ, we call a function v on G *λ-harmonic* if $\Delta v(x) = \lambda v(x)$ for all vertices x.

Note that it follows from formula (1.1) that the Laplacian will be bounded if and only if there exists a constant M such that $m(x) \leq M$ for all vertices x.

Indeed, letting δ_x denote the delta function at a vertex x so that

$$\delta_x(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

we see that the matrix coefficients of the Laplacian are given by

$$\begin{aligned} \Delta(x, y) &= \langle \Delta\delta_x, \delta_y \rangle = \Delta\delta_x(y) \\ &= \begin{cases} m(x) & \text{if } x = y \\ -1 & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As mentioned in the introduction, under the assumption $m(x) \leq M$, all graphs are stochastically complete [6, Theorem 2.10]. Therefore, for the purposes of our inquiry, we do not impose this restriction on the graph and the Laplacian will be an unbounded operator.

Let D be a finite, connected subgraph of G. We then have the following analogue of Green's Theorem.

Lemma 1.2.1.

$$\begin{aligned} \sum_{x \in D} \Delta f(x) g(x) &= \sum_{[x,y] \in \tilde{E}(D)} df([x, y]) dg([x, y]) + \sum_{\substack{x \in D \\ z \sim x, z \notin D}} \big(f(x) - f(z)\big) g(x) \\ &= \sum_{[x,y] \in \tilde{E}(D)} df([x, y]) dg([x, y]) - \sum_{\substack{[x,z] \\ x \in D, z \notin D}} df([x, z]) g(x). \end{aligned}$$

Proof: Every oriented edge $[x, y]$ with $x, y \in V(D)$ contributes two terms to the sum on the left hand side: $\big(f(x) - f(y)\big)g(x)$ from $\Delta f(x)g(x)$ and $\big(f(y) - f(x)\big)g(y)$ from $\Delta f(y)g(y)$. These add up to give

$$\big(f(y) - f(x)\big)\big(g(y) - g(x)\big) = df([x, y])dg([x, y]).$$

The remaining contributions come from any vertex x in D that is connected to a neighbor z which is not in D and these give $\big(f(x) - f(z)\big)g(x) = -df([x, z])g(x)$. □

We say that a vertex x is in the *boundary* of D, and denote this $x \in \partial D$, if it is a vertex of D and is connected to any vertex which is not in D. Otherwise, a vertex x of D is said to be in the *interior* of D, or $x \in \operatorname{int}(D)$. We then see that, if either f or g are zero on the complement of the interior of D, then the second term on the right hand side of the equation above is zero and we can write Lemma 1.2.1 as

$$\langle \Delta f, g\rangle_{V(D)} = \langle df, dg\rangle_{\tilde{E}(D)} = \langle f, \Delta g\rangle_{V(D)}.$$

Also, if f and g are any two functions and one of them is finitely supported, it is true that

$$\langle \Delta f, g\rangle = \langle df, dg\rangle = \langle f, \Delta g\rangle$$

where now the inner products are taken over $V(G)$ and $\tilde{E}(G)$.

Throughout, we wish to study solutions of the *combinatorial heat equation.* These will be functions on G with an additional time parameter in which they are differentiable and which satisfy the equation

$$\Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0$$

for every vertex x and every $t > 0$. We start by recalling a proof of analogues for the weak and strong maximum principles for the heat equation [10, 6].

Lemma 1.2.2. *Suppose that D is a finite, connected subgraph of G and*

$$u : D \times [0,T] \to \mathbf{R}$$

is continuous for $t \in [0,T]$, C^1 for $t \in (0,T)$, and satisfies the combinatorial heat equation:

$$\Delta u + \frac{\partial u}{\partial t} = 0 \text{ on } \operatorname{int} D \times (0,T).$$

Then, if there exists $(x_0,t_0) \in \operatorname{int} D \times (0,T)$ such that (x_0,t_0) is a maximum (or minimum) for u on $D \times [0,T]$, then $u(x,t_0) = u(x_0,t_0)$ for all $x \in D$.

Proof: At either a maximum or minimum, $\frac{\partial u}{\partial t}(x_0, t_0) = 0$, giving that

$$\Delta u(x_0, t_0) = \sum_{x \sim x_0} \Big(u(x_0, t_0) - u(x, t_0) \Big) = 0.$$

In either case, this implies that $u(x, t_0) = u(x_0, t_0)$ for all $x \sim x_0$. Iterating the argument and using the assumption that D is connected gives the statement of the lemma. $\square$

Lemma 1.2.3. *Under the same hypotheses as above we have that*

$$\max_{D \times [0,T]} u = \max_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} u$$

and

$$\min_{D \times [0,T]} u = \min_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} u.$$

Proof: Let $v = u - \epsilon t$ for $\epsilon > 0$. Then $\Delta v + \frac{\partial v}{\partial t} = -\epsilon < 0$. If v has a maximum at $(x_0, t_0) \in \mathrm{int} D \times (0, T]$ then

$$\frac{\partial v}{\partial t}(x_0, t_0) \geq 0 \quad \text{and} \quad \Delta v(x_0, t_0) \geq 0$$

yielding a contradiction. Therefore,

$$\max_{D \times [0,T]} v = \max_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} v.$$

Then

$$\begin{aligned}
\max_{D \times [0,T]} u &= \max_{D \times [0,T]} v + \epsilon t \\
&\leq \max_{D \times [0,T]} v + \epsilon T \\
&= \max_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} v + \epsilon T \\
&\leq \max_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} u + \epsilon T.
\end{aligned}$$

Letting $\epsilon \to 0$ we get that

$$\max_{D \times [0,T]} u = \max_{\substack{D \times \{0\} \ \cup \\ \partial D \times [0,T]}} u.$$

The statement about the minimum follows by applying the argument to $-u$. $\square$

Remark 1.2.1. Using the same techniques as above, it follows that, if u satisfies

$$\Delta u + \frac{\partial u}{\partial t} \geq 0 \textit{ on } \operatorname{int} D \times (0,T)$$

then

$$\min_{D\times[0,T]} u = \min_{\substack{D\times\{0\}\,\cup\\ \partial D\times[0,T]}} u$$

while if u satisfies

$$\Delta u + \frac{\partial u}{\partial t} \leq 0 \textit{ on } \operatorname{int} D \times (0,T)$$

then

$$\max_{D\times[0,T]} u = \max_{\substack{D\times\{0\}\,\cup\\ \partial D\times[0,T]}} u.$$

1.3 Essential Self-Adjointness of the Laplacian

As in the case of the Laplacian on a Riemannian manifold, the Laplacian with domain $C_0(V)$, the set of all finitely supported functions on the graph G, is a symmetric but not self-adjoint operator. It is, however, essentially self-adjoint by which we mean that it has a unique self-adjoint extension $\tilde{\Delta}$ to $\ell^2(V)$, a fact which we prove in this section. Let Δ^* denote the adjoint of Δ with domain $C_0(V)$.

Proposition 1.3.1. *The domain of* Δ^* *is*

$$dom(\Delta^*) = \{f \in \ell^2(V) \mid \Delta f \in \ell^2(V)\}.$$

Proof: By definition

$$\operatorname{dom}(\Delta^*) = \left\{ f \in \ell^2(V) \;\middle|\; \begin{array}{c} \text{there exists a unique } h \in \ell^2(V) \text{ such that} \\ \langle \Delta g, f\rangle = \langle g, h\rangle \text{ for all } g \in C_0(V) \end{array} \right\}$$

and then $\Delta^* f = h$. If g is finitely supported as above then we can apply the analogue of Green's Theorem to get that if $f \in \mathrm{dom}(\Delta^*)$ then

$$\langle \Delta g, f \rangle = \langle g, \Delta f \rangle = \langle g, h \rangle.$$

Letting, $g = \delta_x$ we get that $\Delta f(x) = h(x)$ for all vertices x so that $\Delta f \in \ell^2(V)$. $\square$

Theorem 1.3.1. Δ *with domain* $C_0(V)$ *is essentially self-adjoint.*

Proof: From the criterion stated in [13, Theorem X.26] applied to the operator $(\Delta + I)$ it suffices to show that -1 is not an eigenvalue of Δ^*. In other words, if f satisfies $\Delta^* f = -f$ then f cannot be in $\ell^2(V)$ unless it is exactly 0. As can be seen by applying the analogue of Green's Theorem, pointwise, it is true that $\Delta^* f(x) = \Delta f(x)$. Therefore, if f satisfies $\Delta^* f(x) = -f(x)$ for every vertex x, it follows that

$$\big(m(x) + 1\big) f(x) = \sum_{y \sim x} f(y).$$

Therefore, there must exist a neighbor $y \sim x$ such that $f(y) > f(x)$. By repeating this argument the conclusion follows. $\square$

Chapter 2

The Heat Kernel

2.1 Construction of the Heat Kernel

We now give a construction of the heat kernel $p = p_t(x, y)$ for a infinite, locally finite, connected graph G. By *heat kernel* we mean that $p_t(x, y)$ will be the smallest non-negative function

$$p : V \times V \times \ [0, \infty) \to [0, 1]$$

which is smooth in t, satisfies the heat equation: $\Delta p + \frac{\partial p}{\partial t} = 0$ in either x or y and satisfies: $p_0(x, y) = \delta_x(y)$. The heat kernel will generate a bounded solution of the heat equation on G for any bounded initial condition. That is, for any bounded function u_0, $u(x, t) = \sum_{y \in V} p_t(x, y)u_0(y)$ will give a bounded solution to

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } x \in V, \text{all } t > 0 \\ u(x,0) = u_0(x) & \text{for all } x \in V. \end{cases}$$

The construction given here follows the approach of [3, Section 3] and the formalism of [2].

Starting with an exhaustion sequence of the graph, we construct heat kernels with Dirichlet boundary conditions for each set in the exhaustion. Although we

will exhaust the graph by balls of increasing radii, it will be shown later that the resulting heat kernel is independent of the choice of subgraphs in the exhaustion. Let $x_0 \in V(G)$ be a fixed vertex. We will let $B_r = B_r(x_0)$ denote the ball of radius r about x_0, $\partial B_r = \partial B_r(x_0)$ its boundary, and int B_r, its interior. In particular, if $d(x, x_0)$ denotes the standard metric on graphs,

$$\begin{aligned} V(B_r) &= \{x \in V(G) \mid d(x, x_0) \leq r\} \\ E(B_r) &= \{x \sim y \mid x, y \in V(B_r) \text{ and } x \sim y \in E(G)\}. \end{aligned}$$

We then let $C(B_r, \partial B_r)$ denote functions on B_r which vanish on the boundary ∂B_r and let Δ_r denote the *reduced Laplacian* which acts on these spaces. That is,

$$C(B_r, \partial B_r) = \{f \in C(B_r) \mid f_{|\partial B_r} = 0\}$$

and

$$\Delta_r f(x) = \begin{cases} \Delta f(x) & \text{for } x \in \text{int } B_r \\ 0 & \text{otherwise} \end{cases}$$

for all $f \in C(B_r, \partial B_r)$.

With these definitions we then have:

Lemma 2.1.1. *Δ_r is a self-adjoint, non-negative operator on $C(B_r, \partial B_r)$.*

Proof: This follows from the analogue of Green's Theorem, Lemma 1.2.1, since

$$\langle \Delta_r f, g\rangle_{V(B_r)} = \langle df, dg\rangle_{\tilde{E}(B_r)} = \langle f, \Delta_r g\rangle_{B(B_r)}.$$

$\square$

From Lemma 2.1.1 it follows that all eigenvalues λ_i^r of Δ_r are real and non-negative. In fact, as mentioned in [2], and to be shown later, $\lambda_0(\Delta_r) = \lambda_0^r$, the smallest eigenvalue of Δ_r, is given by the Rayleigh-Ritz quotient:

$$\lambda_0^r = \min_{\substack{f \in C(B_r, \partial B_r) \\ f \not\equiv 0}} \frac{\langle df, df\rangle}{\langle f, f\rangle}$$

so that all of the eigenvalues of Δ_r are positive. Denote by $\{\lambda_i^r\}_{i=0}^{k(r)}$ the set of all eigenvalues of Δ_r listed in increasing order and choose a set $\{\phi_i^r\}_{i=0}^{k(r)}$ of corresponding eigenfunctions which are an orthonormal basis for $C(B_r, \partial B_r)$ with respect to the ℓ^2 inner product. That is, $\{\phi_i^r\}_{0=1}^{k(r)}$ are such that,

$$\Delta_r \phi_i^r = \lambda_i^r \phi_i^r \quad \forall i = 0, \ldots, k(r) \tag{2.1}$$

and

$$\sum_{x \in B_r} \phi_i^r(x) \phi_j^r(x) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We are now ready to define the heat kernels $p_t^r(x, y)$ for each subgraph in the exhaustion.

Definition 2.1.1.

$$p_t^r(x, y) = \sum_{i=0}^{k(r)} e^{-\lambda_i^r t} \phi_i^r(x) \phi_i^r(y) \quad \text{ for all } x, y \in B_r, \text{ all } t \geq 0. \tag{2.3}$$

Theorem 2.1.2. $p_t^r(x, y)$ *has the following properties for every* r*:*

1) $p_t^r(x, y) = p_t^r(y, x)$, $p_t^r(x, y) = 0$ *if either* $x \in \partial B_r$ *or* $y \in \partial B_r$.

2) $\Delta_r p_t^r(x, y) + \frac{\partial}{\partial t} p_t^r(x, y) = 0$ *where* Δ_r *denotes the reduced Laplacian in either* x *or* y.

3) $p_{s+t}^r(x, y) = \sum_{z \in B_r} p_s^r(x, z) p_t^r(z, y)$.

4) $p_0^r(x, y) = \delta_x(y)$ *for* $x, y \in \text{int } B_r$.

5) $p_t^r(x, y) > 0$ *for all* $t > 0$, *all* $x, y \in \text{int } B_r$.

6) $\sum_{y \in B_r} p_t^r(x, y) < 1$ *for all* $t > 0$, *all* $x \in B_r$.

Remark 2.1.3. We could also start by defining the heat semigroup operator as the convergent power series:

$$Q_t^r = e^{-t\Delta_r} = I - t\Delta_r + \frac{t^2}{2}\Delta_r^2 - \frac{t^3}{6}\Delta_r^3 + \ldots$$

and then take its kernel given by $q_t^r(x,y) = \langle Q_t^r \delta_x, \delta_y \rangle = Q_t^r \delta_x(y)$. The equivalence of these two approaches can be seen by applying the maximum principle, Lemma 1.2.3, to the difference of the two kernels.

Proof: **1), 2)** Clear from the definition of $p_t^r(x,y)$, the fact that $\phi_i^r \in C(B_r, \partial B_r)$, that is, $\phi^r_{i|\partial B_r} = 0$, and from (2.1).

3) Using the orthonormality of $\{\phi_i^r\}_{i=1}^{k(r)}$, we compute:

$$\begin{aligned}
\sum_{z\in B_r} p_s^r(x,z)p_t^r(z,y) &= \sum_{z\in B_r}\sum_{i=0}^{k(r)} e^{-\lambda_i^r s}\phi_i^r(x)\phi_i^r(z)\sum_{j=0}^{k(r)} e^{-\lambda_j^r t}\phi_j^r(z)\phi_j^r(y) \\
&= \sum_{i,j=0}^{k(r)} e^{-\lambda_i^r s}e^{-\lambda_j^r t}\phi_i^r(x)\phi_j^r(y)\sum_{z\in B_r}\phi_i^r(z)\phi_j^r(z) \\
&= \sum_{i=0}^{k(r)} e^{-\lambda_i^r (s+t)}\phi_i^r(x)\phi_i^r(y) \qquad (2.2) \\
&= p_{s+t}^r(x,y).
\end{aligned}$$

4) By definition,

$$p_0^r(x,y) = \sum_{i=0}^{k(r)} \phi_i^r(x)\phi_i^r(y).$$

Since $\{\phi_i^r\}_{i=0}^{k(r)}$ form an orthonormal basis, it follows that

$$\delta_x(y) = \sum_{i=0}^{k(r)} \langle \delta_x, \phi_i^r\rangle \phi_i^r(y) = \sum_{i=0}^{k(r)} \phi_i^r(x)\phi_i^r(y).$$

Therefore, $p_0(x,y) = \delta_x(y)$.

5) The maximum principle, Lemma 1.2.3, applied in each of the variables separately to $p_t^r(x,y)$ over the set $B_r \times B_r \times [0,T]$ implies that $0 \le p_t^r(x,y) \le 1$ since $p_0^r(x,y) = \delta_x(y)$ and $p_t^r(x,y) = 0$ if either x or y is in the boundary of B_r.

Now, assume that there exists a $t_0 > 0$ and $\hat{x}, \hat{y} \in \operatorname{int} B_r$ such that $p_{t_0}^r(\hat{x},\hat{y}) = 0$. We may then assume that $(\hat{x},\hat{y},t_0)$ is a minimum for $p_t^r(x,y)$ on $B_r \times B_r \times [0,t_0]$. Then, using the fact that $p_t^r(x,y)$ satisfies the heat equation in both variables and B_r is connected, and applying the argument used in the proof of

Lemma 1.2.2 over the set $B_r \times B_r \times [0, t_0]$ gives:

$$p^r_{t_0}(x, y) = p^r_{t_0}(\hat{x}, \hat{y}) = 0 \quad \text{ for all } x, y \in \text{int } B_r.$$

In particular,

$$p^r_{t_0}(x, x) = \sum_{i=0}^{k(r)} e^{-\lambda^r_i t_0} (\phi^r_i(x))^2 = 0 \quad \text{ for all } x \in \text{int } B_r$$

implying that $\phi^r_i(x) = 0$ for all i and all $x \in \text{int } B_r$ contradicting the fact that $\{\phi^r_i\}_{i=0}^{k(r)}$ forms an orthonormal basis for $C(B_r, \partial B_r)$.

6) We may assume that $x, y \in \text{int } B_r$ since $p^r_t(x, y) = 0$ otherwise. Then, for $x \in \text{int } B_r$ and $t = 0$, we get

$$\sum_{y \in \text{int } B_r} p^r_0(x, y) = \sum_{y \in \text{int } B_r} \delta_x(y) = 1.$$

Now, by using the analogue of Green's Theorem over the interior of B_r, whose boundary consists of vertices in the interior which have a neighbor in the boundary, we will show that the expression $\sum_{y \in \text{int } B_r} p^r_t(x, y)$ is decreasing as a function of t. That is,

$$\begin{aligned}
\frac{\partial}{\partial t} \sum_{y \in \text{int } B_r} p^r_t(x, y) &= \sum_{y \in \text{int } B_r} \frac{\partial}{\partial t} p^r_t(x, y) \\
&= \sum_{y \in \text{int } B_r} -\Delta_y p^r_t(x, y) \\
&= \sum_{\substack{y \in \text{int } B_r \\ z \sim y,\ z \in \partial B_r}} \Big(p^r_t(x, z) - p^r_t(x, y) \Big) \\
&= \sum_{\substack{y \in \text{int } B_r \\ z \sim y,\ z \in \partial B_r}} -p^r_t(x, y) < 0.
\end{aligned}$$

Therefore,

$$\sum_{y \in B_r} p^r_t(x, y) < 1 \quad \text{ for all } t > 0, \text{all } x \in B_r.$$

$\square$

Remark 2.1.4. The proof of Part 6) is identical to the proof of the corresponding property for the Dirichlet heat kernels on a Riemannian manifold [3, Lemma 3.3 Part (i)] and shows that a finite graph with Dirichlet boundary conditions is *not* stochastically complete.

We now wish to show that the $p_t^r(x,y)$ converge to the heat kernel $p_t(x,y)$ mentioned at the beginning of this section. For this purpose the following lemma will be instrumental.

Lemma 2.1.2.

$$p_t^r(x,y) \le p_t^{r+1}(x,y) \quad \textit{for all } t \ge 0, \textit{all } x, y \in B_r.$$

Proof: This is clear for x or y in ∂B_r. Now, fix $y \in \text{int } B_r$ and let

$$u(x,t) = p_t^{r+1}(x,y) - p_t^r(x,y).$$

Then $\Delta u + \frac{\partial u}{\partial t} = 0$ on int $B_r \times (0,T)$ which implies that the minimum of u is attained on the set $(B_r \times \{0\}) \cup (\partial B_r \times [0,T])$. Since, $u(x,0) = 0$ by Part 4) of Theorem 2.1.2 while, on $\partial B_r \times [0,T]$,

$$u(x,t) = p_t^{r+1}(x,y) > 0.$$

It follows that

$$\min_{B_r \times [0,T]} u \ge 0.$$

Therefore, $p_t^{r+1}(x,y) \ge p_t^r(x,y)$ for all $x, y \in B_r$ and for all $t \ge 0$. $\square$

By extending $p_t^r(x,y)$ to be 0 outside of B_r and using $0 \le p_t^r(x,y) \le 1$ and Lemma 2.1.2 we see that $p_t^r(x,y)$ converges pointwise as $r \to \infty$ for all $x, y \in V$ and all $t \ge 0$. Let $p_t(x,y)$ be the limit

$$p_t^r(x,y) \to p_t(x,y) \text{ as } r \to \infty.$$

We will now show that the convergence is uniform in t for every compact interval $[0,T]$. To this end, we fix x and y in V, let $f_r(t) = p_t^r(x,y)$ and $f(t) = p_t(x,y)$.

Then, from the definition and properties of each of the heat kernels p^r, we get that each $f_r : [0,\infty) \to \mathbf{R}$ is C^∞ and satisfies

1) $f_r \le f_{r+1}$

2) $f_r(t) \to f(t)$ pointwise for all t

3) $f_r(t) \le 1$.

Dini's Theorem implies that $f_r \to f$ uniformly on all compact subsets $[0,T] \subset [0,\infty)$.

We will now show that $p_t(x,y)$ satisfies the heat equation. This will follow if we are able to show that $\frac{\partial}{\partial t} p_t^r(x,y)$ converges uniformly in t on compact intervals as $r \to \infty$. But

$$\begin{aligned} \frac{\partial}{\partial t} p_t^r(x,y) &= -\Delta_x p_t^r(x,y) \\ &= \sum_{z \sim x} \big(p_t^r(z,y) - p_t^r(x,y)\big) \end{aligned}$$

and since both $p_t^r(z,y)$ and $p_t^r(x,y)$ converge uniformly in t on $[0,T]$ it follows that $\frac{\partial p}{\partial t}{}_t(x,y)$ exists and is continuous. In fact, iterating this argument and using the fact that $\frac{\partial^i}{\partial t^i} p_t^r(x,y)$ also satisfy the heat equation and are continuous for all i we get that $p_t(x,y)$ is C^∞ in t. Then, from the pointwise convergence of $p_t^r(x,y)$, we get that

$$\begin{aligned} \frac{\partial}{\partial t} p_t(x,y) &= \frac{\partial}{\partial t} \lim_{r\to\infty} p_t^r(x,y) \\ &= \lim_{r\to\infty} \frac{\partial}{\partial t} p_t^r(x,y) \\ &= \lim_{r\to\infty} -\Delta_x p_t^r(x,y) \\ &= -\Delta_x p_t(x,y) \end{aligned}$$

implying that $\Delta_x p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$. The same argument applied in the y variable then gives $\Delta_y p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$. In summation, using the corresponding properties of $p_t^r(x,y)$, Lemma 2.1.2, and what was just shown, we have proved statements 1), 2), 3), 4), 5), and 6) of the following theorem.

Theorem 2.1.5. *$p : V \times V \times [0,\infty) \to \mathbf{R}$ has the following properties:*

1) *$p_t(x,y) > 0$ and $p_t(x,y) = p_t(y,x)$ for all $t > 0$, all $x, y \in V$.*

2) *p is C^∞ in t.*

3) *$\Delta p_t(x,y) + \frac{\partial}{\partial t} p_t(x,y) = 0$ where Δ denotes the Laplacian in either x or y.*

4) *$p_0(x,y) = \delta_x(y)$ for all $x, y \in V$.*

5) *$p_{s+t}(x,y) = \sum_{z\in V} p_s(x,z)p_t(z,y)$.*

6) *$\sum_{y\in V} p_t(x,y) \leq 1$ for all $t \geq 0$, all $x, y \in V$.*

7) *p is independent of the exhaustion used to define it.*

8) *p is the smallest non-negative function that satisfies Properties 3) and 4).*

Proof: **7)** Say D_i is another exhaustion of G. That is, each D_i is a finite and connected subgraph, $D_i \subset D_{i+1}$ for all i, and $G = \bigcup_{i=1}^\infty D_i$. Let $q_t^{D_i}(x,y)$ denote the Dirichlet heat kernels for this exhaustion and say that $q_t^{D_i}(x,y) \to q_t(x,y)$. Then for every D_i there exists R large enough so that $D_i \subset B_R$. By the maximum principle, since $q_t^{D_i}(x,y)$ vanishes on ∂D_i, we obtain $q_t^{D_i}(x,y) \leq p_t^R(x,y)$. Because $p_t^R(x,y) \leq p_t^{R+1}(x,y) \leq \dots$ and $p_t^R(x,y) \to p_t(x,y)$ this implies that $q_t^{D_i}(x,y) \leq p_t(x,y)$. Letting $i \to \infty$ gives

$$q_t(x,y) \leq p_t(x,y).$$

Interchanging the roles of q^{D_i} and p^r in the preceding argument gives $q_t(x,y) \geq p_t(x,y)$ and therefore, $p_t(x,y) = q_t(x,y)$.

8) Say $q_t(x,y)$ is another non-negative function that satisfies Properties 3) and 4). In particular, both q and p^r satisfy the heat equation on int $B_r \times (0,T)$. Since p^r vanishes on ∂B_r while q is non-negative there we get, by applying the

maximum principle to the difference of q and p^r, that $q_t(x,y) \geq p_t^r(x,y)$ on $B_r \times [0,T]$ and hence for all $x, y \in V$ and all $t > 0$. Letting $r \to \infty$ we get that $q_t(x,y) \geq p_t(x,y)$. $\square$

2.2 The Spectral Theorem Construction

As mentioned previously, an alternative way of obtaining the heat kernels $p_t^r(x,y)$ on B_r with Dirichlet boundary conditions is through the convergent power series

$$e^{-t\Delta_r} = I - t\Delta_r + \frac{t^2}{2}\Delta_r^2 - \frac{t^3}{6}\Delta_r^3 + \dots$$

by letting $p_t^r(x,y) = (e^{-t\Delta_r}\delta_x)(y)$. However, on the entire graph, since the Laplacian is not bounded on $\ell^2(V)$, one cannot use the power series approach. One can still construct $e^{-t\tilde{\Delta}}$ for, $\tilde{\Delta}$, the unique self-adjoint extension of Δ to $\ell^2(V)$ by using the functional calculus developed through the spectral theorem [12, Chapter VIII]. The purpose of this section is to show that the construction given in the previous section via exhaustion and this approach result in the same kernel [3, Proposition 4.5]. Let $P_t v(x) = \sum_{y \in V} p_t(x,y)v(y)$ for any bounded function v, with $P_t^r v$ indicating a similar sum for the heat kernel $p_t^r(x,y)$ on B_r. We then have the following theorem which states that P_t and $e^{-t\tilde{\Delta}}$ agree on a dense subset of $\ell^2(V)$ and, as such, have the same kernel:

Theorem 2.2.1.

$$P_t v = e^{-t\tilde{\Delta}} v \quad \textit{for all } v \in C_0(V).$$

Proof: We begin by showing that if $v \in C_0(V)$ then $P_t v \in \ell^2(V)$ and $\Delta P_t v \in \ell^2(V)$. Since v is finitely supported, there exists a ball of large radius R which

contains its support. Therefore,

$$\begin{aligned}
\|P_t^R v\|_{\ell^2(V)}^2 &= \sum_{x\in V}(P_t^R v(x))^2 \\
&= \sum_{x\in V}\left(\sum_{y\in B_R} p_t^R(x,y)v(y)\right)^2 \\
&\leq \sum_{y\in B_R}\left(\sum_{x\in B_R} p_t^R(x,y)^2\right)v(y)^2 \\
&\leq \sum_{y\in B_R} v(y)^2 = \|v\|_{\ell^2(V)}^2.
\end{aligned}$$

By letting $R\to\infty$ and using the dominated convergence theorem it follows that $P_t v\in\ell^2(V)$. In fact, this actually proves that P_t is a bounded operator on $\ell^2(V)$ with $\|P_t\|\leq 1$.

We will now show that $\Delta P_t v = P_t\Delta v$ and since if v is finitely supported then so is Δv it will follows that $\Delta P_t v\in\ell^2(V)$. To show that $\Delta P_t v = P_t\Delta v$ we calculate:

$$\begin{aligned}
\Delta(P_t v)(x) &= \sum_{y\sim x}\big((P_t v)(x)-(P_t v)(y)\big) \\
&= \sum_{y\sim x}\left(\sum_{z\in V}\big(p_t(x,z)-p_t(y,z)\big)v(z)\right).
\end{aligned}$$

Meanwhile, by using the analogue of Green's Theorem and the fact that the heat kernel satisfies the heat equation in both variables, we get

$$\begin{aligned}
P_t(\Delta v)(x) &= \sum_{z\in V} p_t(x,z)\Delta v(z) \\
&= \sum_{z\in V}\Delta_z p_t(x,z)v(z) \\
&= \sum_{z\in V}\Delta_x p_t(x,z)v(z) \\
&= \sum_{z\in V}\left(\sum_{y\sim x}\big(p_t(x,z)-p_t(y,z)\big)\right)v(z).
\end{aligned}$$

We now give the proof of the theorem. Let

$$u(x,t)=\left(P_t-e^{-t\tilde{\Delta}}\right)v(x).$$

Then $u(x,0)=0$ and since $\Delta P_t v$ is in $\ell^2(V)$ we can apply Green's Theorem again to obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\sum_{x\in V} u^2(x,t) &= 2\sum_{x\in V} u(x,t)\frac{\partial}{\partial t}u(x,t) \\
&= -2\sum_{x\in V} u(x,t)\Delta u(x,t) \\
&= -2\sum_{[x,y]\in\tilde{E}} \big(u(y,t)-u(x,t)\big)^2 \le 0
\end{aligned}$$

from which it now follows that $P_t v(x) = e^{-t\tilde{\Delta}}v(x)$ for all finitely supported v. Since both P_t and $e^{-t\tilde{\Delta}}$ are bounded it follows that they are equal on $\ell^2(V)$. $\square$

Chapter 3

Stochastic Incompleteness

3.1 Stochastic Incompleteness

We now define the notion of stochastic incompleteness and recall the proof of the equivalence of several properties and this definition. The material here is adapted from [9, p. 170-172]. We recall the definition of P_t:

$$P_t u_0(x) = \sum_{y \in V} p_t(x, y) u_0(y)$$

for any bounded function u_0 on G. This summation converges from Part 6) of Theorem 2.1.5 and from Part 5), P_t satisfies the semigroup property:

$$P_s(P_t u_0) = P_{s+t} u_0.$$

Apply P_t to the function $\mathbf{1}$ which is exactly 1 on each vertex of G:

$$P_t \mathbf{1}(x) = \sum_{y \in V} p_t(x, y)$$

and note that this sum is less than or equal to 1 from Part 6) of Theorem 2.1.5.

Definition 3.1.1. A graph G is called *stochastically incomplete* if for some

vertex x_0 of G and some $t_0 > 0$

$$P_{t_0}\mathbf{1}(x_0) = \sum_{y \in V} p_{t_0}(x_0, y) < 1.$$

Remark 3.1.2. Although this really is a property of the heat kernel or of the diffusion process which is modeled by the heat kernel, it is customary to say, as above, that it is a property of the underlying space.

Theorem 3.1.3. *The following statements are equivalent:*

1) *For some* $t_0 > 0$, *some* $x_0 \in V$, $P_{t_0}\mathbf{1}(x_0) < 1$.

1') *For all* $t > 0$, *all* $x \in V$, $P_t\mathbf{1}(x) < 1$.

2) *There exists a positive (equivalently, non-zero) bounded function* v *on* G *such that* $\Delta v = \lambda v$ *for any* $\lambda < 0$.

2') *There exists a positive, (equivalently, non-zero) bounded function* v *on* G *such that* $\Delta v \leq \lambda v$ *for any* $\lambda < 0$.

3) *There exists a nonzero, bounded solution to*

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } x \in V, \text{all } t > 0 \\ u(x,0) = 0 & \text{for all } x \in V. \end{cases}$$

Definition 3.1.4. Any function v on G such that $\Delta v = \lambda v$ is called λ-*harmonic* whereas if $\Delta v \leq \lambda v$, v is called λ-*subharmonic.*

Therefore, stochastic incompleteness is equivalent to the existence of a positive, bounded λ-harmonic (or λ-subharmonic) function for negative λ and to the non-uniqueness of bounded solutions for the heat equation on G.

Proof:

1') $\Rightarrow$ 1) Obvious.

1) $\Rightarrow$ 1') If there exists $x_0 \in V$ and a $t_0 > 0$ such that $P_{t_0}\mathbf{1}(x_0) = 1$ then by the stong maximum principle for the heat equation, Lemma 1.2.2, applied to the function $P_t\mathbf{1}$ we get that

$$P_{t_0}\mathbf{1}(x) = 1 \text{ for all } x.$$

Now, if $s < t_0$ then it follows from the semigroup property that

$$P_{t_0}\mathbf{1} = P_{t_0-s}(P_s\mathbf{1}) \leq P_{t_0-s}\mathbf{1} \leq 1.$$

For any $t_0 > 0$ such that $P_{t_0}\mathbf{1} = 1$ it follows that the inequalities become equalities and, in particular, $P_s\mathbf{1} = 1$ for all $s < t_0$. If $s > t_0$ then there exists a k such that $s < kt_0$ and by the semigroup property we get that

$$P_{kt_0}\mathbf{1} = (\underbrace{P_{t_0}\dots P_{t_0}}_{k})\mathbf{1} = 1$$

provided that $P_{t_0}\mathbf{1} = 1$ giving $P_s\mathbf{1} = 1$ from the same argument as above.

1') $\Rightarrow$ 2) For any $\lambda < 0$, let $w(x) = \int_0^\infty e^{\lambda t}u(x,t)dt$ where $u(x,t) = P_t\mathbf{1}(x) < 1$ by assumption. Then

$$\begin{aligned} 0 < w &< \int_0^\infty e^{\lambda t}dt \\ &= \frac{1}{\lambda}\left(e^{\lambda t}\Big|_0^\infty\right) \\ &= \frac{1}{\lambda}(0-1) = -\frac{1}{\lambda}. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} \Delta w = \int_0^\infty e^{\lambda t}\Delta u(x,t)dt &= -\int_0^\infty e^{\lambda t}\frac{\partial u}{\partial t}(x,t)dt \\ &= -e^{\lambda t}u(x,t)\Big|_0^\infty + \int_0^\infty \lambda e^{\lambda t}u(x,t)dt \\ &= 1 + \lambda w. \end{aligned}$$

If $v = 1 + \lambda w$, then v satisfies

$$\Delta v = \lambda\Delta w = \lambda(1+\lambda w) = \lambda v$$

which shows that v is λ-harmonic. Since $0 < w < -\frac{1}{\lambda}$ we have $0 < v < 1$ so that v is positive and bounded.

2) $\Rightarrow$ **2')** Clear.

2') $\Rightarrow$ **2)** Exhaust the graph G by finite, connected subgraphs D_i. That is, $D_i \subset D_{i+1}$ and $G = \bigcup_{i=0}^{\infty} D_i$ where each D_i is finite and connected. Let Δ_i denote the reduced Laplacian acting on the space $C(D_i, \partial D_i)$ of functions on D_i which vanish on the boundary ∂D_i. Then, for $\lambda < 0$, one can solve

$$\begin{cases} \Delta_i v_i = \lambda v_i \quad \text{on int } D_i \\ v_{i|\partial D_i} = 1. \end{cases} \tag{3.1}$$

Indeed, letting $\mathbf{1}_{D_i}$ denote the function that is 1 on every vertex of D_i and 0 elsewhere, if v_i is a solution to the above then $w_i = v_i - \mathbf{1}_{D_i}$ would vanish on the boundary of D_i and on the interior would satisfy

$$\Delta_i w_i(x) = \Delta_i v_i(x) = \lambda v_i(x) = \lambda(w_i(x) + 1).$$

That is,

$$(\Delta_i - \lambda I) w_i = \lambda_{\text{int } D_i}$$

where $\lambda_{\text{int } D_i}$ denotes the function that is equal to λ on every vertex in the interior of D_i and is 0 on ∂D_i. Since $\lambda < 0$, $\Delta_i - \lambda I$ is invertible on $C(D_i, \partial D_i)$ and so

$$w_i = (\Delta_i - \lambda I)^{-1}(\lambda_{\text{int } D_i})$$

yielding

$$v_i = (\Delta_i - \lambda I)^{-1}(\lambda_{\text{int } D_i}) + \mathbf{1}_{D_i}$$

as a solution for (3.1).

We now claim that

$$0 < v_i \leq 1 \text{ on } D_i. \tag{3.2}$$

This follows from the fact that, if there exists an x_0 in the interior of D_i such that $v_i(x_0) \le 0$, then we may assume that x_0 is a minimum for v_i and

$$\Delta v_i(x_0) = \sum_{x \sim x_0} \big(v_i(x_0) - v_i(x)\big) \le 0$$

while $\Delta v_i(x_0) = \lambda v_i(x_0) \ge 0$ so that $\Delta v_i(x_0) = 0$. This implies that $v_i(x) = v_i(x_0)$ for all neighbors of x_0 and by repeating the argument we would get that v_i is a non-positive constant on D_i contradicting that $v_i = 1$ on the boundary of D_i.

Therefore, $v_i > 0$ and so $\Delta v_i < 0$ on the interior which implies that

$$\max_{D_i} v_i = \max_{\partial D_i} v_i = 1.$$

Indeed, at an interior maximum $\Delta v_i(x_0) \ge 0$. This completes the proof of (3.2).

Furthermore, if we extend each v_i to be exactly 1 outside of D_i, it is true that

$$v_i \ge v_{i+1}.$$

This is clear on ∂D_i since $v_i = 1$ there while $v_{i+1} \le 1$. On the interior of D_i we have that $\Delta(v_i - v_{i+1}) = \lambda(v_i - v_{i+1})$ from which it follows that $v_i - v_{i+1} > 0$ by the same argument that gives $v_i > 0$ above.

The v_i therefore, form a non-increasing, bounded sequence so that

$$v_i \to v$$

where $0 \le v \le 1$ and $\Delta v = \lambda v$. What remains to be shown is that v is positive (or non-zero) and here we use the assumption that there exists on G a positive (or non-zero), bounded λ-subharmonic function w. That is, w is positive (or non-zero), bounded and satisfies $\Delta w \le \lambda w$ on G. Assuming that $w \le 1$, we show that

$$v_i \ge w \text{ on } D_i \text{ for all } i.$$

This can be seen as follows: on the interior of D_i,

$$\Delta(v_i - w) \ge \lambda(v_i - w). \tag{3.3}$$

Therefore, if there exists an x_0 in the interior of D_i such that $(v_i - w)(x_0) < 0$ and x_0 is a minimum for $v_i - w$ then by computation $\Delta(v_i - w)(x_0) \leq 0$ while $\Delta(v_i - w)(x_0) \geq \lambda(v_i - w)(x_0) > 0$ from (3.3). The contradiction implies that $v_i \geq w$ on D_i and by passing to the limit we get that $v \geq w$ so that v is either positive or non-zero depending on w.

2)$\Rightarrow$ 3) Let $w(x,t) = e^{-\lambda t}v(x)$ where v is a positive, bounded λ-harmonic function for $\lambda < 0$. Then w is positive and bounded on $V \times [0,T]$, and satisfies $\Delta w + \frac{\partial w}{\partial t} = 0$ with $w(x,0) = v(x)$. The function $P_t v$ also satisfies both equations and, moreover,

$$\sup_{x \in V} P_t v(x) \leq \sup_{x \in V} v(x) \tag{3.4}$$

while

$$w(x,t) > v(x) \text{ for all } t > 0, \text{ all } x \in V. \tag{3.5}$$

Therefore, we have two different bounded solutions to

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } (x,t) \in V \times (0,T) \\ u(x,0) = v(x) & \text{all } x \in V. \end{cases}$$

Taking the difference of the two solutions gives a nonzero, bounded solution to

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } (x,t) \in V \times (0,T) \\ u(x,0) = 0 & \text{all } x \in V. \end{cases} \tag{3.6}$$

Therefore, we have shown the existence of a nonzero, bounded solution for the heat equation with initial condition 0 for a finite time interval. In the argument below, we show that this is enough to imply condition 1), that is, $P_t\mathbf{1} < 1$. Then, given $P_t\mathbf{1} < 1$, $\mathbf{1} - P_t\mathbf{1}$ will give a nonzero, bounded solution to (3.6) for an infinite time interval, completing the proof.

We also note that the assumption that v is positive was not essential for the argument as can be seen by putting norms about $P_t v$, v, and w in (3.4) and (3.5). Therefore, the existence of any bounded, *non-zero* λ-harmonic function

will imply stochastic incompleteness and then the argument giving the implication $1') \Rightarrow 2)$ shows that there then exists a *positive*, bounded λ-harmonic function on G.

3) $\Rightarrow$ 1) Suppose that $u(x,t)$ is nonzero, bounded and satisfies

$$\begin{cases} \Delta u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } (x,t) \in V \times (0,T) \\ u(x,0) = 0 & \text{all } x \in V. \end{cases}$$

Then, by rescaling, we may assume that $|u(x,t)| < 1$ for all x and t, and that there exists an x_0 and $t_0 > 0$ such that $u(x_0,t_0) > 0$. Then, $w(x,t) = 1 - u(x,t)$ is bounded, positive and satisfies

$$\begin{cases} \Delta w(x,t) + \frac{\partial w}{\partial t}(x,t) = 0 & \text{for all } (x,t) \in V \times (0,T) \\ w(x,0) = 1 & \text{all } x \in V. \end{cases} \tag{3.7}$$

Furthermore, $w(x_0,t_0) < 1$. Now, letting $P_t^r\mathbf{1}(x) = \sum_{y\in B_r} p_t^r(x,y)$, and applying the maximum principle for the heat equation to $P_t^r\mathbf{1}(x) - w(x,t)$ on $B_r \times [0,T]$, we get that $P_t^r\mathbf{1}(x) < w(x,t)$ for all r. Therefore, letting $r \to \infty$,

$$P_{t_0}\mathbf{1}(x_0) \leq w(x_0,t_0) < 1.$$

$\square$

3.2 Model Trees

We now turn our focus to a family of particular trees and study under what conditions they are stochastically complete. A tree will be called *model* if it contains a vertex x_0, henceforth called the *root* for the model, such that the valence $m(x)$ is constant on spheres $S_r(x_0) = S_r$ of radius r about x_0. That is, if

$$S_r = \{x \mid d(x,x_0) = r\}$$

then

$$m(x) = m(r) \text{ for all } x \in S_r.$$

For $r > 0$, we let $n(r) = m(r) - 1$ denote the *branching* of T, that is, the number of edges connecting a vertex in S_r with vertices in S_{r+1}, and let $n(0) = m(x_0)$. We then denote such trees T_n to indicate that their structure is completely encoded in the branching function $n(r)$.

The main purpose of this section is to prove the following theorem which tells us precisely when such trees are stochastically complete.

Theorem 3.2.1. *T_n is stochastically complete if and only if*

$$\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty.$$

The idea of the proof is to study positive λ-harmonic functions on T_n for $\lambda < 0$. By averaging over spheres we can reduce to the case of λ-harmonic functions depending only on the distance r from the root x_0. It then turns out that such a function will be bounded if and only if the series above converges. Since the existence of a positive, bounded, λ-harmonic function is equivalent to stochastic completeness, Theorem 3.2.1 will then follow.

Let, therefore, $v(r)$ denote a function on the vertices of T_n depending only on $r = r(x) = d(x, x_0)$, the distance from a vertex to the root. When the Laplacian is applied to such a function we get

$$\begin{aligned} \Delta v(r) &= \big(n(r)+1\big)v(r) - n(r)v(r+1) - v(r-1) \\ &= n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big). \end{aligned}$$

We will study the existence and boundedness of such functions on T_n when, in addition, they are positive and λ-harmonic for a negative λ, that is, they satisfy $\Delta v(r) = \lambda v(r)$ for $\lambda < 0$. We start by showing that there is no loss in generality in restricting our study of λ-harmonic functions to only functions of this type.

Lemma 3.2.1. *If there exists a positive, bounded λ-harmonic function on T_n then there exists one depending only on the distance from x_0.*

Proof: Let $u(x)$ denote a positive, bounded λ-harmonic function on T_n. If S_r denotes the sphere of radius r about the root x_0 and $\mathrm{Vol}(S_r)$ denotes its volume, that is,

$$\mathrm{Vol}(S_r) = \#\{x \mid x \in S_r\}$$

then we define a function $v(r)$ depending only on the radius r by averaging u over S_r:

$$v(r) = \frac{1}{\mathrm{Vol}(S_r)} \sum_{x \in S_r} u(x).$$

Clearly, such a function will be positive and bounded since $u(x)$ is. We now show that $v(r)$ is also λ-harmonic. At x_0, since $\mathrm{Vol}(S_1) = n(0)$, we have:

$$\begin{aligned}
\Delta v(0) &= n(0)\Big(v(0) - v(1)\Big) \\
&= n(0)\left(u(x_0) - \frac{1}{n(0)} \sum_{x \in S_1} u(x)\right) \\
&= n(0)u(x_0) - \sum_{x \sim x_0} u(x) \\
&= \Delta u(x_0) = \lambda u(x_0) = \lambda v(0).
\end{aligned}$$

Now, if $x \in S_r$ for $r > 0$, then

$$\begin{aligned}
\Delta u(x) &= \big(n(r)+1\big)u(x) - \sum_{\substack{z \sim x \\ z \in S_{r+1}}} u(z) - u(y) \\
&= \lambda u(x)
\end{aligned}$$

where y is the unique neighbor of x that is in S_{r-1}. Therefore,

$$\big(n(r) + 1 - \lambda\big)u(x) = \sum_{\substack{z \sim x \\ z \in S_{r+1}}} u(z) + u(y). \tag{3.8}$$

We will average this equation over S_r and use the following version of the formula $n(r) \cdot \mathrm{Vol}(S_r) = \mathrm{Vol}(S_{r+1})$ which relates the volume of spheres of different radius in T_n:

$$\frac{1}{\mathrm{Vol}(S_r)} = \frac{n(r)}{\mathrm{Vol}(S_{r+1})}. \tag{3.9}$$

From the definition of v, we get

$$\begin{aligned}\big(n(r)+1-\lambda\big)v(r) &= \frac{(n(r)+1-\lambda)}{\mathrm{Vol}(S_r)}\sum_{x\in S_r}u(x)\\ &= \frac{1}{\mathrm{Vol}(S_r)}\sum_{x\in S_r}\left(\sum_{\substack{z\sim x\\ z\in S_{r+1}}}u(z)+\sum_{\substack{y\sim x\\ y\in S_{r-1}}}u(y)\right) \qquad (3.8)\\ &= \frac{1}{\mathrm{Vol}(S_r)}\sum_{z\in S_{r+1}}u(z)+\frac{n(r-1)}{\mathrm{Vol}(S_r)}\sum_{y\in S_{r-1}}u(y)\\ &= \frac{n(r)}{\mathrm{Vol}(S_{r+1})}\sum_{z\in S_{r+1}}u(z)+\frac{1}{\mathrm{Vol}(S_{r-1})}\sum_{y\in S_{r-1}}u(y) \qquad (3.9)\\ &= n(r)v(r+1)+v(r-1)\end{aligned}$$

or precisely that $\Delta v(r)=\lambda v(r)$. □

Therefore, on a model tree, the existence of any positive, bounded λ-harmonic function is equivalent to the existence of such a function depending only on the distance to the root. The values of such a function are determined by the value of the function at the root and are given by:

$$v(1) = \left(1-\frac{\lambda}{n(0)}\right)v(0) \qquad (3.10)$$

$$v(r+1) = \frac{1}{n(r)}\left(\big(n(r)+1-\lambda\big)v(r)-v(r-1)\right). \qquad (3.11)$$

We will now study under what conditions such a function will remain bounded. We start by showing that such a function must increase with the radius.

Lemma 3.2.2. *If $v>0$ satisfies $\Delta v=\lambda v$ for $\lambda<0$ then*

$$v(r)<v(r+1) \text{ for all } r\geq 0.$$

Proof: The proof is by induction. We have that $v(0)<v(1)$ from (3.10).

Now, assuming that $v(r-1)<v(r)$,

$$\Delta v(r)=n(r)\big(v(r)-v(r+1)\big)+\big(v(r)-v(r-1)\big)=\lambda v(r)$$

gives

$$n(r)\Big(v(r) - v(r+1)\Big) = \lambda v(r) - \Big(v(r) - v(r-1)\Big) < 0$$

implying

$$v(r) < v(r+1).$$

$\square$

Lemma 3.2.3. *If* $\Delta v = \lambda v$ *with* $v > 0$ *and* $\lambda < 0$ *then*

$$\prod_{i=0}^{r}\left(1 - \frac{\lambda}{n(i)}\right) v(0) < v(r+1) < \prod_{i=0}^{\infty}\left(1 + \frac{1-\lambda}{n(i)}\right) v(0).$$

Consequently, for λ negative, a positive, λ-harmonic function on T_n depending only on the distance from the root remains bounded if and only if $\prod_{i=0}^{\infty}\left(1 + \frac{1}{n(i)}\right) < \infty$. That is, the following conditions are equivalent:

1) $v(r)$ is bounded

2) $\prod_{i=0}^{\infty}\left(1 + \frac{1}{n(i)}\right) < \infty$

3) $\sum_{i=0}^{\infty} \frac{1}{n(i)} < \infty$.

Proof: For the upper bound, we rewrite the relation $\Delta v(r) = \lambda v(r)$ as:

$$\Big(n(r) + 1 - \lambda\Big)v(r) - n(r)v(r+1) = v(r-1) > 0.$$

Therefore,

$$\Big(n(r) + 1 - \lambda\Big)v(r) > n(r)v(r+1)$$

or

$$\begin{aligned} v(r+1) &< \frac{(n(r) + 1 - \lambda)}{n(r)} v(r) \\ &= \left(1 + \frac{1-\lambda}{n(r)}\right) v(r). \end{aligned}$$

Now, iterate this relation down to $v(0)$:

$$\begin{aligned}
v(r+1) &< \left(1 + \frac{1-\lambda}{n(r)}\right) v(r) \\
&< \left(1 + \frac{1-\lambda}{n(r)}\right)\left(1 + \frac{1-\lambda}{n(r-1)}\right) v(r-1) \\
&< \prod_{i=0}^{r} \left(1 + \frac{1-\lambda}{n(i)}\right) v(0) \\
&< \prod_{i=0}^{\infty} \left(1 + \frac{1-\lambda}{n(i)}\right) v(0).
\end{aligned}$$

For the lower bound, we use Lemma 3.2.2, which implies that $v(r) - v(r-1) > 0$ as follows:

$$\begin{aligned}
\Delta v(r) &= n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\
&> n(r)\Big(v(r) - v(r+1)\Big).
\end{aligned}$$

Since $\Delta v(r) = \lambda v(r)$, this gives

$$n(r)\Big(v(r) - v(r+1)\Big) < \lambda v(r)$$

or

$$\left(1 - \frac{\lambda}{n(r)}\right) v(r) < v(r+1).$$

Iterating as before gives

$$\prod_{i=0}^{r} \left(1 - \frac{\lambda}{n(i)}\right) v(0) < v(r+1)$$

completing the proof of the lemma. $\square$

Proof of Theorem 3.2.1: By Theorem 3.1.3, stochastic incompleteness is equivalent to the existence of a positive, bounded λ-harmonic function for $\lambda < 0$. We can define such a function on T_n depending only on the distance from x_0 by (3.10) and (3.11). If $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$, this function will remain bounded by Lemma 3.2.3.

Now, if $\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty$ then every positive, λ-harmonic function depending only on the radius from the root will be unbounded. Therefore, every positive, λ-harmonic function on T_n will be unbounded by Lemma 3.2.1 so that T_n is stochastically complete. $\square$

Remark 3.2.2. We would like to point out the relationship between Theorem 3.2.1 and the case of *spherically symmetric* or *model* manifolds on which we base our definition of model trees. M_σ, a Riemannian manifold of dimension d with pole o, is called *model* if

i) topologically, $M_\sigma \setminus \{o\}$ is the product of an open interval I and the sphere S^{d-1}. Therefore, each point $x \in M_\sigma \setminus \{o\}$ can be identified with a pair (r, θ) where $r \in I$ and $\theta \in S^{d-1}$.

ii) the metric on M_σ is given by

$$ds^2 = dr^2 + \sigma^2(r) d\theta^2 \tag{3.12}$$

where $d\theta^2$ denotes the standard Euclidean metric on S^{d-1}. Here, σ is a smooth, positive function on I sometimes called the *twisting* or *warping function* [9, p. 145-148].

It follows from (3.12) that the area of a sphere of radius r in M_σ is given by

$$A(S_r) = \omega_d \sigma^{d-1}(r) \tag{3.13}$$

where ω_d is the area of the unit sphere in $\mathbf{R}^d$. Now, it is shown in [9, Corollary 6.8] that a geodesically complete, noncompact, model manifold is stochastically complete if and only if

$$\int^{\infty} \frac{\mathrm{Vol}(B_r)}{A(S_r)} dr = \infty$$

where $\mathrm{Vol}(B_r)$ denotes the Riemannian volume of the geodesic ball in M_σ. For example, if, for large r, $A(S_r) \leq e^{r^2}$ then M_σ is stochastically complete, whereas, if $A(S_r) = e^{r^{2+\epsilon}}$ for any positive ϵ, then M_σ will be incomplete.

Observe from (3.13) that, on M_σ,

$$\begin{aligned} dA(S_r) &= \omega_d(d-1)\sigma^{d-2}(r)\sigma'(r)\ dr \\ &= (d-1)\omega_d(\ln\sigma(r))' A(S_r)\ dr \end{aligned}$$

so that

$$\frac{dA(S_r)}{A(S_r)} = c(\ln\sigma(r))'\ dr \tag{3.14}$$

for $c = (d-1)\ \omega_d$.

Meanwhile, for model trees, temporarily using the notation $A(S_r) = \mathrm{Vol}(S_r)$ and $dA(S_r) = A(S_{r+1}) - A(S_r)$, it follows that

$$dA(S_r) = \big(n(r) - 1\big)A(S_r)$$

so that

$$\frac{dA(S_r)}{A(S_r)} = \big(n(r) - 1\big). \tag{3.15}$$

Therefore, comparing (3.14) and (3.15) we see that $n(r)$ and $(\ln\sigma(r))'$ play a similar role and the correspondence between the borderlines for stochastic completeness is exact.

3.3 Comparison Theorems

Throughout this section, we assume that T_n denotes a model tree with root vertex x_0 while T denotes a general tree. From Theorem 3.2.1 in the last section, we know that T_n will be stochastically complete if and only if $\sum_{r=0}^{\infty}\frac{1}{n(r)} = \infty$ and we wish to obtain a similar criterion for T. In order to state our results for T we make the following definitions.

Definition 3.3.1. For a vertex $x_0 \in T$, let

$$\begin{aligned} \underline{m}(r) = \underline{m}_{x_0}(r) &= \min_{x\in S_r(x_0)} m(x) \\ M(r) = M_{x_0}(r) &= \max_{x\in S_r(x_0)} m(x). \end{aligned}$$

The following result is an immediate consequence of our criterion for model trees and the characterization of stochastic incompleteness in terms of λ-subharmonic functions.

Theorem 3.3.2. *Assume that* $T_n \subseteq T$ *and that* $\underline{m}_{x_0}(r) = \underline{m}(r) = \min_{x \in S_r(x_0)} m(x)$ *satisfies*

$$n(r) \leq \underline{m}(r) - 1 \textit{ for all } r > 0.$$

Then, if T_n *is stochastically incomplete, so is* T.

Proof: Since T_n is stochastically incomplete, there exists a bounded, positive function $v(r)$ on T_n such that $v(r) < v(r+1)$ and $\Delta v(r) = \lambda v(r)$ for $\lambda < 0$. Let $r(x) = d(x, x_0)$ be the distance between x_0 and $x \in T$ and define a function u on T by

$$u(x) = v(r(x)).$$

Clearly, $u(x)$ will be bounded and positive since v is. Now, it follows from the inequalities $v(0) - v(1) < 0$ and $n(0) \leq m(x_0)$ that $u(x)$ is λ-subharmonic at $x = x_0$:

$$\begin{aligned} \Delta u(x_0) &= m(x_0)u(x_0) - \sum_{x \sim x_0} u(x) \\ &= m(x_0)\Big(v(0) - v(1)\Big) \\ &\leq n(0)\Big(v(0) - v(1)\Big) \\ &= \lambda v(0) = \lambda u(x_0). \end{aligned}$$

Now, suppose that $r(x) = r > 0$ and y denotes the unique neighbor of x in S_{r-1}. Then, since $n(r) \leq m(x) - 1$,

$$\begin{aligned} \Delta u(x) &= m(x)u(x) - \sum_{\substack{z \sim x \\ z \in S_{r+1}}} u(z) - u(y) \\ &= (m(x) - 1)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\ &\leq n(r)\Big(v(r) - v(r+1)\Big) + \Big(v(r) - v(r-1)\Big) \\ &= \lambda v(r) = \lambda u(x). \end{aligned}$$

Thus, u is a positive, bounded λ-subharmonic function on T implying that T is stochastically incomplete. $\square$

This result has the following corollary:

Corollary 3.3.1. *If T is a tree with a vertex x_0 such that $\underline{m}_{x_0}(r) = \underline{m}(r) = \min_{x \in S_r(x_0)} m(x)$ satisfies $\underline{m}(r) > 1$ and*

$$\sum_{r=0}^{\infty} \frac{1}{\underline{m}(r)} < \infty$$

then T is stochastically incomplete.

Proof: From the assumption on T, we can embed $T_n \subseteq T$, where T_n is defined by

$$n(r) = \underline{m}(r) - 1 \text{ for } r > 0$$

and $n(0) = m(x_0)$. Then $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$ giving that T_n is stochastically incomplete and so is, therefore, T. $\square$

Remark 3.3.3. This theorem and its corollary are unsatisfactory in the sense that they require the tree to grow very rapidly in all directions from x_0 in order to be stochastically incomplete. However, as we will see in Theorem 3.4.1 in the next section, it is sufficient that the tree grows very rapidly in some direction from x_0.

We now prove the inverse of the last result for a general graph G.

Theorem 3.3.4. *If G is any graph with a vertex x_0 such that $M_{x_0}(r) = M(r) = \max_{x \in S_r(x_0)} m(x)$ satisfies*

$$\sum_{r=0}^{\infty} \frac{1}{M(r)} = \infty$$

then G is stochastically complete.

Proof: Let u be a positive, λ-harmonic function on G for $\lambda < 0$. We will show that under the assumption on G, u must be unbounded. At x_0, the relation

$\Delta u(x_0) = \lambda u(x_0)$, gives that

$$\sum_{x \sim x_0} u(x) = \big(m(x_0) - \lambda\big) u(x_0). \tag{3.16}$$

This implies that there exists $x_1 \sim x_0$ such that

$$u(x_1) \geq \left(1 - \frac{\lambda}{m(x_0)}\right) u(x_0).$$

If not, then for all $x \sim x_0$, $u(x) < \left(1 - \frac{\lambda}{m(x_0)}\right) u(x_0)$, giving that

$$\sum_{x \sim x_0} u(x) < m(x_0) \left(1 - \frac{\lambda}{m(x_0)}\right) u(x_0)$$

contradicting (3.16).

Now, by repeating the argument at x_1, we get that there must exist a neighbor $y \sim x_1$ such that

$$u(y) \geq \left(1 - \frac{\lambda}{m(x_1)}\right) u(x_1).$$

Although y is not necessarily in $S_2(x_0)$ we can repeat the argument until we obtain a vertex $x_2 \in S_2(x_0)$ such that

$$\begin{aligned} u(x_2) &\geq \left(1 - \frac{\lambda}{m(x_1)}\right) u(x_1) \\ &\geq \left(1 - \frac{\lambda}{m(x_1)}\right) \left(1 - \frac{\lambda}{m(x_0)}\right) u(x_0). \end{aligned}$$

Iterating this argument, we get a sequence of distinct vertices $x_0 \sim x_1 \sim x_2 \sim \ldots$ such that $x_r \in S_r(x_0)$ and

$$u(x_r) \geq \prod_{i=0}^{r-1} \left(1 - \frac{\lambda}{m(x_i)}\right) u(x_0).$$

Since $\sum_{i=0}^{\infty} \frac{1}{m(x_i)} \geq \sum_{i=0}^{\infty} \frac{1}{M(i)} = \infty$ implies that $\prod_{i=0}^{\infty} \left(1 - \frac{\lambda}{m(x_i)}\right) = \infty$ it follows that u cannot remain bounded, giving that G must be stochastically complete. $\square$

Remark 3.3.5. Theorem 3.3.4 is a significant improvement over the result mentioned in the introduction [6, Theorem 2.10] which states that the same

conclusion as above holds if the valence is bounded above by a constant. In fact the proof there can be extended to show that if $M(r)$ is $o(r)$ then the graph is stochastically complete whereas our result says that $M(r)$ can even be $O(r)$.

A corollary of Theorem 3.3.4 for trees is the following:

Corollary 3.3.2. *Assume that $T \subseteq T_n$ with $x_0 \in T$. If T_n is stochastically complete then so is T.*

Proof: Since T_n is stochastically complete we have that $\sum_{r=0}^{\infty} \frac{1}{n(r)} = \infty$ implying $\sum_{r=0}^{\infty} \frac{1}{M(r)} = \infty$ so that T is stochastically complete. $\square$

3.4 General Trees

The purpose of this section is to follow-up on the remark following Corollary 3.3.1. The result there states that a general tree T will be stochastically incomplete if, starting out at a fixed vertex, the branching grows rapidly in all directions. The next theorem states that the same conclusion holds if the branching grows rapidly in just one direction.

We start by slightly altering the notation used in the previous section. If x_0 and x_1 are vertices of T with $x_0 \sim x_1$ then we now denote

$$\underline{m}(r) = \underline{m}_{\{x_0,x_1\}}(r) = \min_{\substack{x \in S_r(x_0) \\ d(x,x_1)=r-1}} m(x) \quad \text{for } r \geq 1$$

so that the minimum is now taken over those x in $S_r(x_0)$ such that $d(x,x_1) = r-1$.

Theorem 3.4.1. *If T is a tree with a vertex $x_0 \in T$ such that for some $x_1 \sim x_0$, $\underline{m}(r) = \underline{m}_{\{x_0,x_1\}}(r)$ satisfies $\underline{m}(r) > 1$ and*

$$\sum_{r=1}^{\infty} \frac{1}{\underline{m}(r)} < \infty$$

then T is stochastically incomplete.

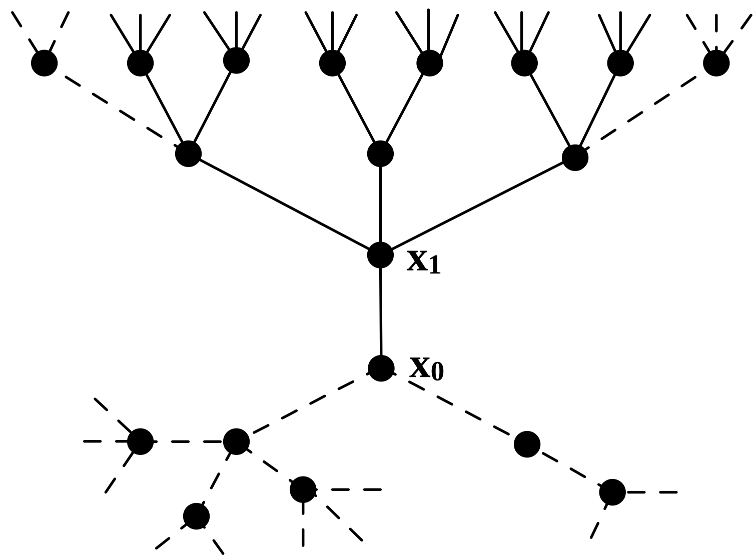

Figure 3.1: T with T_n in solid.

The proof of the theorem will use the following general proposition.

Proposition 3.4.1. *For a graph G with $x_0 \in G$ and any $\lambda < 0$ there exists a function v on G such that $v(x_0) = 1$, $0 < v(x) \leq 1$ for all vertices x and $\Delta v(x) = \lambda v(x)$ for all vertices $x \neq x_0$.*

Proof of Theorem 3.4.1: Assuming Proposition 3.4.1, we apply it to define a positive, bounded λ-harmonic function v on the part of T below x_0 in Figure 3.1 with $v(x_0) = 1$. For the part of T above x_0 in Figure 3.1, the assumption on T implies that we can embed a stochastically incomplete model subtree T_n with root vertex x_0 where $n(0) = 1$ and $n(r) = \underline{m}(r) - 1$ for $r \geq 1$. We extend the function v to be defined on T_n by first making it λ-harmonic at x_0:

$$v(x_1) = \big(m(x_0) - \lambda\big) - \sum_{\substack{x \sim x_0 \\ x \neq x_1}} v(x).$$

Then, if r denotes the distance to the root, v can be defined on the rest of T_n by

$$v(r+1) = \left(1 + \frac{1-\lambda}{n(r)}\right) v(r) - \frac{1}{n(r)} v(r-1) \quad \text{for } r \geq 2.$$

This function will remain bounded since $\sum_{r=1}^{\infty} \frac{1}{n(r)} < \infty$. Therefore, by the argument used in the proof of Theorem 3.3.2, there exists a positive, bounded λ-subharmonic function on T. By Theorem 3.1.3, T is then stochastically incomplete. $\square$

Proof of Proposition 3.4.1: Let $B_r(x_0)$ denote the ball of radius r about x_0 in G. For $\lambda < 0$, on each $B_r(x_0)$ there exists a unique solution to the following system of equations:

$$\begin{cases} \Delta v_r(x) = \lambda v_r(x) & \text{for all } x \in \text{int } B_r \setminus \{x_0\} \\ v_r(x_0) = 1 & \\ v_r(x) = 0 & \text{for all } x \in \partial B_r. \end{cases} \tag{3.17}$$

Indeed, from basic linear algebra, since the system has the same number of equations as there are values for v_r, there will exist a unique solution if 0 is the only function which satisfies

$$\begin{cases} \Delta v_r(x) = v_r(x) & \text{for all } x \in \text{int } B_r \setminus \{x_0\} \\ v_r(x_0) = 0 & \\ v_r(x) = 0 & \text{for all } x \in \partial B_r. \end{cases} \tag{3.18}$$

To show that this is so, suppose that v_r is a non-zero solution to (3.18). We can then assume that there exists a vertex $\hat{x}$ in the interior such that $v_r(\hat{x}) > 0$ and $\hat{x}$ is a maximum for v_r on $B_r(x_0)$. Then, by calculation, $\Delta v_r(\hat{x}) \geq 0$, while $\Delta v_r(\hat{x}) = \lambda v_r(\hat{x}) < 0$ giving a contradiction. The same argument could be used to show that v_r cannot have a negative minimum. Therefore, any solution to (3.18) must be zero. This gives existence and uniqueness of a solution to (3.17).

Therefore, for $\lambda < 0$, for each r there exists a unique solution to

$$\begin{cases} \Delta v_r(x) = \lambda v_r(x) & \text{for all } x \in \text{int } B_r \setminus \{x_0\} \\ v_r(x_0) = 1 & \\ v_r(x) = 0 & \text{for all } x \in \partial B_r. \end{cases}$$

On the interior of B_r, a solution must satisfy

$$0 < v_r \leq 1.$$

For, supposing that there exists a vertex $\hat{x}$ in the interior such that $v_r(\hat{x}) \leq 0$ and $\hat{x}$ is a minimum for v_r then, as before, by calculation, $\Delta v_r(\hat{x}) \leq 0$ while $\Delta v_r(\hat{x}) = \lambda v_r(\hat{x}) \geq 0$. Therefore, $v_r(x) = v_r(\hat{x})$ for all x next to $\hat{x}$ and, by repeating the argument, it would follow that v_r is a constant function for a non-positive constant yielding a contradiction since $v_r(x_0) = 1$. Hence, $v_r > 0$ implying $\Delta v_r < 0$ for all vertices in the interior except for x_0 so that $v_r \leq 1$ since at an interior maximum $\Delta v_r \geq 0$.

Similarly, since $\Delta(v_{r+1} - v_r) = \lambda(v_{r+1} - v_r)$, it follows that $v_r \leq v_{r+1}$ on B_r and by extending each v_r to be 0 outside of B_r, we get that

$$v_r \leq v_{r+1} \text{ on } G.$$

Therefore, we can define v as the limit

$$v_r \to v \text{ as } r \to \infty.$$

It follows that v satisfies $0 < v \leq 1$, $v(x_0) = 1$, and $\Delta v = \lambda v$ for all vertices of G except for x_0. □

3.5 Heat Kernel Comparison

The purpose of this section is to prove two theorems which compare the heat kernel on a general tree to the heat kernel on a model. These theorems were inspired by an analogous result of Cheeger and Yau on model manifolds [1, Theorem 3.1]. Fixing a vertex x_0 in a tree T, we now denote

$$\underline{m}(r) = \min_{x \in S_r(x_0)} m(x) \quad \text{and} \quad M(r) = \max_{x \in S_r(x_0)} m(x)$$

the minimum and maximum valence along the spheres $S_r(x_0)$. Throughout, we use the notation $\rho_t(x_0, x)$ for the heat kernel on T_n, while $p_t(x_0, x)$ will denote

the heat kernel on T. We will first show that, as a function of x, $\rho_t(x_0, x)$ is constant on the spheres $S_r(x_0)$ in T_n. Let $\rho_t(r) = \rho_t(0, r(x))$ denote this common value. Then the two main theorems of the section can be stated as follows:

Theorem 3.5.1. *If $M(r) \leq n(r) + 1$ for all $r > 0$ then*

$$\rho_t(r) \leq p_t(x_0, x)$$

for all $x \in S_r(x_0) \subset T$.

Theorem 3.5.2. *If $n(r) \leq \underline{m}(r) - 1$ for all $r > 0$ then*

$$p_t(x_0, x) \leq \rho_t(r)$$

for all $x \in S_r(x_0) \subset T$.

The proofs will follow easily from the maximum principle for the heat equation once we establish two general lemmas concerning the heat kernel on T_n. We start by proving the property of the heat kernel mentioned at the start of this section.

Lemma 3.5.1. *On T_n*

$$\rho_t(x_0, x) = \rho_t(r)$$

for all $x \in S_r(x_0)$.

Proof: This result is essentially a restatement of the fact that the coefficients of the Laplacian depend only on the valence which, on T_n, only depends on the distance from the root.

We establish the result for the heat kernels $\rho_t^R(x_0, x)$ on $B_R(x_0)$ with Dirichlet boundary conditions and pass to the limit. The heat kernel $\rho_t^R(x_0, x)$ is the kernel of the operator semigroup

$$e^{-t\Delta_R} = I - t\Delta_R + \frac{t^2\Delta_R^2}{2} - \frac{t^3\Delta_R^3}{6} + \ldots$$

where Δ_R denotes the reduced Laplacian on $B_R(x_0)$. That is,

$$\rho_t^R(x_0,x) = \langle \delta_{x_0}, \delta_x \rangle - t\Delta_R(x_0,x) + \frac{t^2}{2}\Delta_R^2(x_0,x) - \dots$$

where the coefficients of the Laplacian $\Delta_R(x_0,x)$ are given by

$$\begin{aligned} \Delta_R(x_0,x) &= \Delta_R \delta_{x_0}(x) \\ &= \begin{cases} n(0) & \text{if } x = x_0 \\ -1 & \text{if } x \in S_1(x_0) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\Delta_R^{m+n}(x_0,x) = \sum_{y \in B_R} \Delta_R^m(x_0,y)\Delta_R^n(y,x).$$

Therefore, these only depend on the distance between x_0 and x. □

Lemma 3.5.2. *On T_n*

$$\rho_t(r) \geq \rho_t(r+1) \quad \textit{for all } r \geq 0.$$

Proof: We start with the general fact that for any graph

$$\frac{\partial}{\partial t} p_t(x,x) \leq 0 \quad \text{for all } t > 0.$$

Indeed, working with the heat kernels $p_t^R(x,y)$ on $B_R(x_0)$ with Dirichlet boundary conditions but now using the eigenfunction expansion we have

$$p_t^R(x,x) = \sum_{j=0}^{k(R)} e^{-\lambda_j^R t}\big(\phi_j^R(x)\big)^2$$

implying

$$\frac{\partial}{\partial t} p_t^R(x,x) = \sum_{j=0}^{k(R)} -\lambda_j^R e^{-\lambda_j^R t}\big(\phi_j^R(x)\big)^2 < 0$$

since $\lambda_j^R > 0$. By passing to the limit we get that $\frac{\partial}{\partial t} p_t(x,x) \leq 0$.

Therefore, in particular, $\frac{\partial}{\partial t}\rho_t^R(0) < 0$ which implies that

$$\Delta \rho_t^R(0) = n(0)\Big(\rho_t^R(0) - \rho_t^R(1)\Big) > 0.$$

Thus,

$$\rho_t^R(0) > \rho_t^R(1).$$

We also have that

$$\rho_t^R(R-1) > \rho_t^R(R) = 0$$

so that, as a function of r, $\rho_t^R(r)$ is decreasing at $r = 0$ as well as at $r = R - 1$ for all $t > 0$.

Fix now a time $t_0 > 0$ and assume that there exists an $i_0 > 0$ such that

$$\rho_{t_0}^R(i_0) < \rho_{t_0}^R(i_0 + 1)$$

where i_0 is the smallest number with this property. Therefore, as a function of r, $\rho_{t_0}^R(r)$ achieves a local minimum at $r = i_0$. Since $\rho_{t_0}^R(r)$ is decreasing again at $r = R - 1$, it follows that there must exist a j_0, $i_0 < j_0 \leq R - 1$ such that

$$\rho_{t_0}^R(j_0) > \rho_{t_0}^R(j_0 + 1).$$

Therefore, $\rho_{t_0}^R(r)$ has a local maximum at $r = j_0$. It follows from calculation that

$$\Delta\rho_{t_0}^R(i_0) < 0 \quad \text{and} \quad \Delta\rho_{t_0}^R(j_0) > 0$$

implying

$$\frac{\partial}{\partial t}\rho_{t_0}^R(i_0) > 0 \quad \text{and} \quad \frac{\partial}{\partial t}\rho_{t_0}^R(j_0) < 0.$$

Therefore, at a previous time $t_1 < t_0$, $\rho_{t_1}^R(r)$, as a function of r, achieves a smaller minimum than $\rho_{t_0}^R(i_0)$ and a larger maximum then $\rho_{t_0}^R(j_0)$. That is, there exist i_1 and j_1 such that

$$\rho_{t_1}^R(i_1) < \rho_{t_0}^R(i_0) \text{ and } \rho_{t_1}^R(j_1) > \rho_{t_0}^R(j_0)$$

implying, in particular, that

$$\rho_{t_1}^R(i_1) < \rho_{t_1}^R(j_1).$$

Since this argument can be repeated for any positive time t, it follows that eventually we reach $t = 0$ at which point we would have an i and j, $0 < i < j < R$, such that

$$\rho_0^R(i) < \rho_0^R(j)$$

contradicting the fact that $p_0^R(r) = 0$ for all $r \neq 0$. Therefore, $\rho_t^R(r) \geq \rho_t^R(r+1)$ for all r and letting $R \to \infty$ we get

$$\rho_t(r) \geq \rho_t(r+1).$$

$\square$

Proof of Theorems 3.5.1 and 3.5.2: The proofs of the the two Theorems are nearly identical so we give details for the proof of Theorem 3.5.1 and then point out the modifications needed for the proof of Theorem 3.5.2. We will denote the Laplacian on T_n by Δ_{T_n} to distinguish it from Δ_T, the Laplacian on T, wherever it is necessary.

For Theorem 3.5.1 it follows from the assumption $M(r) \leq n(r) + 1$ that we can embed T into T_n. Hence, we think of $T \subseteq T_n$, and want to show that $\rho_t(r(x)) \leq p_t(x_0, x)$ for all vertices of T, where $r(x) = d(x, x_0)$. We work with the Dirichlet heat kernels on $B_R(x_0) \subset T$ and consider the function

$$u^R(x,t) = \rho_t^R(r(x)) - p_t^R(x_0, x)$$

on $B_R \times [0, S]$. Then

$$u^R(x, 0) = 0 \text{ for all } x \in B_R$$

and

$$u^R(x,t) = 0 \text{ for all } x \in \partial B_R, \text{ all } t.$$

Furthermore, it follows from $m(x) - 1 \leq n(r(x))$ and from Lemma 3.5.2 that

$\rho_t^R(r(x))$ satisfies the following inequality for any $x \in B_R$:

$$\begin{aligned}
\Delta_T \rho_t^R(r(x)) &= (m(x)-1)\Big(\rho_t^R(r(x)) - \rho_t^R(r(x)+1)\Big) \\
&\quad + \rho_t^R(r(x)) - \rho_t^R(r(x)-1) \\
&\leq n(r(x))\Big(\rho_t^R(r(x)) - \rho_t^R(r(x)+1)\Big) \\
&\quad + \rho_t^R(r(x)) - \rho_t^R(r(x)-1) \\
&= \Delta_{T_n}\rho_t^R(r(x)) = -\frac{\partial}{\partial t}\rho_t^R(r(x)).
\end{aligned}$$

Therefore, $u^R(x,t)$ satisfies

$$\Delta_T u^R(x,t) + \frac{\partial}{\partial t}u^R(x,t) \leq 0.$$

By applying the maximum principle for the heat equation (see Remark 1.2.1), it follows that

$$\max_{B_R\times[0,S]} u^R(x,t) = \max_{\substack{B_R\times\{0\}\ \cup \\ \partial B_R\times[0,S]}} u^R(x,t) = 0.$$

Therefore, $\rho_t^R(r(x)) - p_t^R(x,x_0) \leq 0$ so that $\rho_t^R(r(x)) \leq p_t^R(x,x_0)$. Since this holds for every R, by letting $R \to \infty$, we get that

$$\rho_t(r(x)) \leq p_t(x,x_0).$$

For Theorem 3.5.2, from the assumption that $n(r) \leq \underline{m}(r) - 1$, we may assume that $T_n \subseteq T$ and we want to show that $p_t(x_0,x) \leq \rho_t(r(x))$. First, extend ρ to be defined on all of T as before by letting:

$$\rho_t(x) = \rho_t(r(x)) \text{ for } x \in T.$$

Then, since $n(r) \leq m(x) - 1$ for all $x \in S_r \subset T$, it follows that

$$u^R(x,t) = \rho_t^R(r(x)) - p_t^R(x_0,x)$$

now satisfies

$$\Delta_T u^R(x,t) + \frac{\partial}{\partial t}u_t^R(x,t) \geq 0.$$

This implies that

$$\min_{B_R\times[0,S]} u^R(x,t) = \min_{\substack{B_R\times\{0\}\ \cup \\ \partial B_R\times[0,S]}} u^R(x,t) = 0$$

implying

$$p_t^R(x_0,x) \leq \rho_t^R(r(x)).$$

$\square$

In fact, using the same proof as above, we can extend the result of Theorem 3.5.2 to a slightly more general graph G in which a model subtree T_n may be embedded. Essentially, we obtain G by allowing any two vertices on the same sphere to be connected by an edge in T from Theorem 3.5.2. To make this precise, we introduce the following notation which will be useful later as well. Let $x_0 \in G$ be a fixed vertex and $r(x) = d(x,x_0)$.

Definition 3.5.3. For $x \in G$ let

$$\begin{aligned} m_0(x) &= \#\{y \mid y\sim x \text{ and } r(y)=r(x)\} \\ m_{+1}(x) &= \#\{y \mid y\sim x \text{ and } r(y)=r(x)+1\} \\ m_{-1}(x) &= \#\{y \mid y\sim x \text{ and } r(y)=r(x)-1\} \end{aligned}$$

That is, $m_0(x), m_{+1}(x)$, and $m_{-1}(x)$ denote the number of vertices that are the same distance, further away, and closer to x_0 than is x, respectively. We then state and prove the following:

Theorem 3.5.4. *If G is any graph with $n(r) \leq m_{+1}(x)$ for all $x \in S_r(x_0)$ and $m_{-1}(x) = 1$ for all vertices of G then*

$$p_t(x_0,x) \leq \rho_t(r)$$

for all $x \in S_r(x_0) \subset G$.

Proof: As before, we may assume that $T_n \subset G$. Extend ρ to be defined on G by letting:

$$\rho_t(x) = \rho_t(r(x)) \text{ for } x \in G.$$

] Then, for $x \in S_r(x_0) \subseteq G$,

$$\begin{aligned}
\Delta_G \rho_t(r) &= m_0(x)\big(\rho_t(r) - \rho_t(r)\big) + \ m_{+1}(x)\big(\rho_t(r) - \rho_t(r+1)\big) \\
&\quad + m_{-1}(x)\big(\rho_t(r) - \rho_t(r-1)\big) \\
&= m_{+1}(x)\big(\rho_t(r) - \rho_t(r+1)\big) + \ \rho_t(r) - \rho_t(r-1) \\
&\geq \Delta_{T_n}\rho_t(r) = -\frac{\partial}{\partial t}\rho_t(r).
\end{aligned}$$

The rest of the proof is identical to the proof of Theorem 3.5.2. □

3.6 Bounded Laplacian

In this section, we introduce the bounded Laplacian Δ_{bd} and prove that, with this operator, any graph is stochastically complete. That is, in particular, bounded solutions to the heat equation involving Δ_{bd} with bounded initial conditions are unique. We refer to [4, 5] for the definitions involved.

We define the bounded Laplacian to be the operator

$$\begin{aligned}
\Delta_{bd} f(x) &= f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y) \\
&= \frac{1}{m(x)} \Delta f(x).
\end{aligned}$$

In order for the analogue of Green's Theorem to hold we alter the inner product on the space of functions on the graph. We now let

$$\langle f, g \rangle_{bd} = \sum_{x \in V} f(x)g(x)m(x)$$

while keeping the inner product on edges the same as before. It is now true that

$$\langle \Delta_{bd} f, g \rangle_{bd} = \langle df, dg \rangle$$

for all f such that

$$\sum_{x \in V} f(x)^2 m(x) < \infty.$$

What distinguishes Δ_{bd} from Δ is that it is a bounded operator without the assumption $m(x) \leq M$ necessary to imply that Δ is bounded. This can be seen as follows.

$$\begin{aligned} \langle df, df \rangle &= \sum_{[x,y] \in \tilde{E}} \big(f(y) - f(x)\big)^2 \\ &\leq 2 \sum_{[x,y] \in \tilde{E}} \big(f^2(y) + f^2(x)\big) \\ &= 2 \sum_{x \in V} f^2(x) m(x) \;=\; 2\langle f, f \rangle_{bd}. \end{aligned}$$

Therefore, $\|d\| \leq \sqrt{2}$ implying $\|\Delta_{bd}\| \leq 2$.

We next prove that any graph is stochastically complete with respect to this operator by studying λ-harmonic functions of Δ_{bd}.

Theorem 3.6.1. *If v is a positive function on G satisfying*

$$\Delta_{bd} v(x) = \lambda v(x)$$

for $\lambda < 0$ then v will be unbounded.

In particular, since the proof of the equivalence of the various formulations of stochastic incompleteness for Δ holds for Δ_{bd}, we have that

Corollary 3.6.1. *For any bounded function u_0, the bounded solution to*

$$\begin{cases} \Delta_{bd} u(x,t) + \frac{\partial u}{\partial t}(x,t) = 0 & \text{for all } x \in V, \text{all } t > 0 \\ u(x,0) = u_0(x) & \text{for all } x \in V \end{cases}$$

is unique.

Proof of Theorem 3.6.1: The proof is essentially the proof of Theorem 3.3.4 rewritten for the bounded Laplacian. Fix a vertex x_0 of G. We will show that there exists a sequence of distinct vertices

$$x_0 \sim x_1 \sim x_2 \sim \dots$$

such that

$$v(x_i) \to \infty \text{ as } i \to \infty.$$

At x_0,

$$\Delta_{bd} v(x_0) = v(x_0) - \frac{1}{m(x_0)} \sum_{x \sim x_0} v(x) = \lambda v(x_0)$$

implies that

$$\sum_{x \sim x_0} v(x) = m(x_0)(1 - \lambda) v(x_0). \tag{3.19}$$

Therefore, there exists a neighbor x_1 of x_0 such that

$$v(x_1) \geq (1 - \lambda) v(x_0).$$

Since, if not, if $v(x) < (1 - \lambda) v(x_0)$ for all $x \sim x_0$, then

$$\sum_{x \sim x_0} v(x) < m(x_0)(1 - \lambda) v(x_0)$$

contradicting (3.19). Applying the argument now at x_1 we get a neighbor x_2 of x_1 such that

$$v(x_2) \geq (1 - \lambda) v(x_1) \geq (1 - \lambda)^2 v(x_0).$$

In general, we get a sequence of distinct vertices $x_0 \sim x_1 \sim x_2 \sim \ldots$ such that

$$v(x_i) \geq (1 - \lambda)^i v(x_0)$$

implying that

$$v(x_i) \to \infty \text{ as } i \to \infty.$$

□

Chapter 4

Spectral Analysis

4.1 Bottom of the Spectrum

We recall the definition of $\lambda_0(\Delta)$, the bottom of the spectrum of the Laplacian on a general graph G, and prove a characterization of it in terms of λ-harmonic functions. This result was inspired by an analogous result in [14, Theorem 2.1] and was proven for the bounded Laplacian in [4].

Fix a vertex x_0 in G and let $B_r = B_r(x_0)$ denote the ball of radius r about x_0 with boundary ∂B_r as before. Also, let Δ_r denote the reduced Laplacian acting on the space $C(B_r, \partial B_r)$ of functions on B_r that vanish on the boundary ∂B_r. We define then $\lambda_0^r = \lambda_0(\Delta_r)$ as

$$\lambda_0^r = \lambda_0(\Delta_r) = \min_{\substack{f \in C(B_r, \partial B_r) \\ f \not\equiv 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$$

and show, as in [2, Lemma 1.9], that

Lemma 4.1.1. *λ_0^r is the smallest eigenvalue of Δ_r. Furthermore, if f_0 is a function in $C(B_r, \partial B_r)$ such that*

$$\lambda_0^r = \frac{\langle df_0, df_0 \rangle}{\langle f_0, f_0 \rangle} \tag{4.1}$$

then $\Delta_r f_0 = \lambda_0^r f_0$ and f_0 can be chosen so that $f_0 > 0$ on the interior of B_r.

Proof: If λ is any eigenvalue of Δ_r with eigenfunction f then

$$\frac{\langle df, df\rangle}{\langle f, f\rangle} = \frac{\langle \Delta_r f, f\rangle}{\langle f, f\rangle} = \lambda$$

implies that $\lambda \geq \lambda_0^r$.

Now, if f_0 satisfies (4.1) above and $\{\lambda_i^r\}_{i=0}^{k(r)}$ are the eigenvalues of Δ_r with $\{\phi_i^r\}_{i=0}^{k(r)}$ a set of corresponding eigenfunctions which are an orthonormal basis for $C(B_r, \partial B_r)$ then

$$f_0 = \sum_{i=0}^{k(r)} a_i \phi_i^r$$

where $a_i = \langle f_0, \phi_i^r\rangle$. We wish to show that $a_i = 0$ if $\lambda_i^r \neq \lambda_0^r$. This can be seen as follows

$$\begin{aligned}
0 \leq & \left\langle d\Big(f_0 - \sum_{i=0}^{k(r)} a_i \phi_i^r\Big), d\Big(f_0 - \sum_{j=0}^{k(r)} a_j \phi_j^r\Big)\right\rangle \\
=& \langle df_0, df_0\rangle - 2\sum_{i=0}^{k(r)} a_i \langle f_0, \Delta_r \phi_i^r\rangle + \sum_{i,j=0}^{k(r)} a_i a_j \langle \phi_i^r, \Delta_r \phi_j^r\rangle \\
=& \langle df_0, df_0\rangle - 2\sum_{i=0}^{k(r)} a_i^2 \lambda_i^r + \sum_{i,j=0}^{k(r)} a_i a_j \lambda_j^r \langle \phi_i^r, \phi_j^r\rangle \\
=& \langle df_0, df_0\rangle - \sum_{i=0}^{k(r)} a_i^2 \lambda_i^r
\end{aligned}$$

implies that

$$\langle df_0, df_0\rangle \ \geq \ \sum_{i=0}^{k(r)} a_i^2 \lambda_i^r.$$

While (4.1) gives

$$\langle df_0, df_0\rangle = \lambda_0^r \langle f_0, f_0\rangle = \lambda_0^r \sum_{i=0}^{k(r)} a_i^2.$$

Therefore, $a_i = 0$ if $\lambda_i^r \neq \lambda_0^r$.

Now, noting that

$$\langle f_0, f_0\rangle = \langle |f_0|, |f_0|\rangle$$

while

$$\langle df_0, df_0\rangle \ \geq \ \langle d|f_0|, d|f_0|\rangle$$

it is clear that (4.1) can only be decreased by replacing f_0 by $|f_0|$ and we may assume at the onset that $f_0 \geq 0$. Then, if there exists a vertex $\hat{x}$ in the interior of B_r where $f_0(\hat{x}) = 0$ then it follows from $\Delta_r f_0 = \lambda_0^r f_0$ that

$$\Delta_r f_0(\hat{x}) = -\sum_{x \sim \hat{x}} f_0(x) = 0.$$

Therefore, $f_0(x) = 0$ for all $x \sim \hat{x}$. Repeating this argument would give that $f_0 = 0$ on the interior of B_r yielding a contradiction. $\square$

It follows from Lemma 4.1.1 that

$$\lambda_0^r \geq \lambda_0^{r+1} > 0$$

so we may define

$$\lambda_0 = \lambda_0(\Delta) = \lim_{r \to \infty} \lambda_0(\Delta_r).$$

Remark 4.1.1. It is clear that this number is independent of the choice of exhaustion sequence for the graph G since if $\{D_i\}_{i=0}^{\infty}$ is any other exhaustion sequence then for each R there is I_R large enough so that $B_R \subset D_{I_R}$. Therefore, by Lemma 4.1.1, $\lambda_0^R \geq \lambda_0^{D_{I_R}}$. Reversing the roles of B_r and D_i, we get that $\lambda_0^{B_r}$ and $\lambda_0^{D_i}$ converge to the same number. Also, for future reference, we point out that $\lambda_0(\Delta)$ can also be defined as

$$\lambda_0(\Delta) = \inf_{\substack{f \in C_0(V) \\ f \not\equiv 0}} \frac{\langle df, df \rangle}{\langle f, f \rangle}$$

where $C_0(V)$ denotes the set of finitely supported functions on the graph G.

We now state and prove the following characterization of $\lambda_0(\Delta)$ in terms of λ-harmonic functions [14, Theorem 2.1].

Theorem 4.1.2. *For every $\lambda \leq \lambda_0(\Delta)$ there exists a positive λ-harmonic function. For every $\lambda > \lambda_0(\Delta)$ there is no such function.*

Proof: The proof of the first part is a variation of the argument given in [2, Theorem 2.4]. See also [4, Proposition 1.5] for the case of the bounded Laplacian and [8, Lemma 1] for manifolds.

We start with the case of $\lambda = \lambda_0(\Delta)$. From Lemma 4.1.1, for each r there exists a positive function v_r such that

$$\Delta_r v_r = \lambda_0^r v_r \text{ on int } B_r.$$

We normalize this function so that $v_r(x_0) = 1$ and extend it to be 0 outside of B_r. We will show that this function is bounded for all vertices x. Let $M(i) = M_{x_0}(i) = \max_{x \in S_i(x_0)} m(x)$ as before. Then if $x_1 \in S_1 \subset \text{int } B_r$ it follows from $\Delta v_r(x_0) > 0$ that

$$m(x_0)v_r(x_0) > \sum_{x \sim x_0} v_r(x) \geq v_r(x_1)$$

implying

$$v_r(x_1) < M(0).$$

By repeating the same argument we get that if $x_i \in S_i$ where $i < r$ then

$$v_r(x_i) < M(i-1)M(i-2)\ldots M(0). \tag{4.2}$$

Now, using the diagonal process we can find a subsequence of $\{v_r\}_{r=1}^{\infty}$ which converges for all vertices x. Denote this as

$$v_{r_k}(x) \to v(x) \text{ as } k \to \infty \text{ for all } x.$$

It follows that $\Delta v(x) = \lambda_0 v(x)$ for all vertices x, $v \geq 0$ and $v(x_0) = 1$. By using the same argument as in the proof of Lemma 4.1.1 if there exists an x where $v(x) = 0$ then v would have to be constantly 0 yielding a contradiction since $v(x_0) = 1$. This completes the case of $\lambda = \lambda_0$.

For the case of $\lambda < \lambda_0(\Delta)$ we modify the argument above as follows. First, as noted in the proof of the implication $2') \Rightarrow 2)$ in Theorem 3.1.3, since $\lambda < \lambda_0(\Delta) \leq \lambda_0(\Delta_r)$ the operator $(\Delta_r - \lambda I)$ is positive and hence invertible on

$C(B_r, \partial B_r)$, the space of all functions on B_r which vanish on ∂B_r, so that one can find a function v_r which satisfies

$$\begin{cases} \Delta_r v_r = \lambda v_r & \text{on int } B_r \\ v_{r|\partial B_r} = 1. \end{cases}$$

Indeed, as before, one can let $v_r = (\Delta_r - \lambda I)^{-1}(\lambda_{\text{int } B_r}) + \mathbf{1}$, where $\lambda_{\text{int } B_r}$ is the function equal to λ on every vertex in the interior of B_r and 0 elsewhere and $\mathbf{1}$ is equal to 1 on every vertex of B_r. We renormalize v_r so that it is equal to 1 at x_0 and call it u_r, that is, let

$$u_r = \frac{1}{v_r(x_0)} v_r.$$

Now, if $u_r \geq 0$ on the interior of B_r then $u_r > 0$ on the interior of B_r by the same argument as above. To show that $u_r \geq 0$ we assume that there exists a vertex $\hat{x}$ in the interior of B_r where $u_r(\hat{x}) < 0$ and let w be a new function defined by

$$w(x) = \begin{cases} u_r(x) & \text{for } x \text{ such that } u_r(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $w(x) = 0$ if x is in the boundary of B_r, so the error term vanishes when we apply the analogue of Green's Theorem to w below. Now, if x is a vertex in the interior of B_r such that $u_r < 0$ for x and all neighbors of x then $\Delta w(x) = \Delta u_r(x)$. If x is a vertex where $u_r(x) < 0$ and x has a neighbor y for which $u_r(y) \geq 0$ then $\Delta w(x) \geq \Delta u_r(x)$. Combining these, it follows that

$$\begin{aligned} \langle dw, dw \rangle &= \langle \Delta w, w \rangle \\ &= \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} \Delta w(x) w(x) \\ &\leq \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} \Delta u_r(x) u_r(x) \\ &= \lambda \sum_{\substack{x \in \text{int } B_r \\ u_r(x) < 0}} u_r(x)^2 \\ &= \lambda \langle w, w \rangle \end{aligned}$$

so that

$$\frac{\langle dw, dw \rangle}{\langle w, w \rangle} \leq \lambda.$$

From Lemma 4.1.1 it would follow that $\lambda_0(\Delta_r) \leq \lambda$ contradicting the assumption that $\lambda < \lambda_0(\Delta)$. Therefore, $u_r \geq 0$ on the interior of B_r and so $u_r > 0$ there as well.

Now, if $0 < \lambda < \lambda_0(\Delta)$ then we can use the same argument as above to show that $u_r(x)$ is bounded for all vertices x as in (4.2). If $\lambda \leq 0$ then from $\Delta u_r = \lambda u_r$ the bound becomes

$$u_r(x_1) \leq M(0) - \lambda$$

for all $x_1 \in S_1$ and

$$u_r(x_i) \leq (M(i-1) - \lambda)(M(i-2) - \lambda) \ldots (M(0) - \lambda)$$

for all $x_i \in S_i$. The remainder of the argument is the same as before. This completes the proof of the first part of Theorem 4.1.2.

The proof of the second part of Theorem 4.1.2 is adapted from [14, p. 761]. See [4] for a different proof involving the use of Green's Theorem. Suppose that there exists a positive function v such that $\Delta v = \lambda v$. Then, letting

$$u(x,t) = e^{-\lambda t} v(x)$$

and

$$w(x,t) = \sum_{y \in B_r} p_t^r(x,y) v(y)$$

we see that both u and w satisfy the heat equation on int $B_r \times (0, S)$ and $u(x,0) = w(x,0) = v(x)$ on int B_r with $u(x,0) = v(x) > 0$ and $w(x,0) = 0$ on ∂B_r. By applying the maximum principle for the heat equation to the difference of the two functions we get that

$$\min_{B_r \times [0,S]} (u - w) = \min_{\substack{B_r \times \{0\} \cup \\ \partial B_r \times [0,S]}} (u - w) \geq 0$$

by what was noted above and since $w(x,t)$ vanishes on ∂B_r while $u(x,t)$ is positive there. Therefore, $u(x,t) \geq w(x,t)$ or

$$e^{-\lambda t} v(x) \geq \sum_{y \in B_r} p_t^r(x,y) v(y) \text{ on } B_r \times [0,S].$$

By using the eigenvalue and eigenfunction expansion for $p_t^r(x,y)$ we now get that

$$\begin{aligned} v(x) &\geq \sum_{y \in B_r} e^{\lambda t} p_t^r(x,y) v(y) \\ &= \sum_{y \in B_r} e^{\lambda t} \sum_{i=0}^{k(r)} e^{-\lambda_i^r t} \phi_i^r(x) \phi_i^r(y) v(y) \\ &= \sum_{y \in B_r} \sum_{i=0}^{k(r)} e^{(\lambda - \lambda_i^r)t} \phi_i^r(x) \phi_i^r(y) v(y) \end{aligned}$$

and if $\lambda > \lambda_0^r$ then the right hand side would tend to ∞ as $t \to \infty$. It follows that $\lambda \leq \lambda_0^r$ for all r so that $\lambda \leq \lambda_0(\Delta)$. □

4.1.1 Relationship to Stochastic Incompleteness

In light of the large volume growth that is required for a graph to be stochastically incomplete and the well-known relationship between the bottom of the spectrum and Cheeger's constant which is, at least partially, outlined in this subsection and the next section, it might seem plausible to conjecture that stochastic incompleteness would imply that $\lambda_0(\Delta) > 0$. The purpose of this subsection is to give an example where this is not the case.

The example is constructed as follows: start with a model tree T_n which is stochastically incomplete, that is, such that $\sum_{r=0}^{\infty} \frac{1}{n(r)} < \infty$, and attach to the root vertex x_0 an infinitely long path $x_0 \sim x_1 \sim x_2 \sim \ldots$. The resulting tree will be stochastically incomplete by Theorem 3.4.1.

We now show that $\lambda_0(\Delta) = 0$. As noted before

$$\lambda_0 \leq \frac{\langle df, df \rangle}{\langle f, f \rangle} \tag{4.3}$$

for any nonzero, finitely supported function f. By taking any finite subgraph D and substituting its characteristic function 1_D into (4.3) we get that

$$\lambda_0 \leq \frac{L(\partial D)}{\mathrm{Vol}(D)}$$

where

$$L(\partial D) = \#\{y \sim x \mid x \in D \text{ and } y \notin D\}$$

that is, the number of edges with one vertex in D and one not in D, and $\mathrm{Vol}(D)$ denotes the number of vertices in D. By taking increasingly larger connected subgraphs of the path that was added onto out model tree T_n, it is clear that this ratio goes to 0 since $L(\partial D) = 2$, for all such subgraphs.

4.2 Lower Bounds

In this section we prove some estimates of $\lambda_0(\Delta)$, the bottom of the spectrum of Δ. In order to prove our results, we work with the bounded Laplacian Δ_{bd} and use the characterization of $\lambda_0(\Delta_{bd})$ in terms of Cheeger's constant proved in [5] to get a lower bound for $\lambda_0(\Delta_{bd})$. We then transfer this result to obtain a lower bound for $\lambda_0(\Delta)$.

We recall that Δ_{bd} is given by

$$\Delta_{bd} f(x) = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y) = \frac{1}{m(x)} \Delta f(x).$$

The bottom of the spectrum is, as for Δ, given by an exhaustion argument or, equivalently, as

$$\lambda_0(\Delta_{bd}) = \inf_{\substack{f \in C_0(V) \\ f \not\equiv 0}} \frac{\langle \Delta_{bd} f, f \rangle_{bd}}{\langle f, f \rangle_{bd}}$$

where the infimum is taken over all nonzero, finitely supported functions f and the inner product is now given by

$$\langle f, f \rangle_{bd} = \sum_{x \in V} f(x)^2 m(x).$$

For a finite subgraph D we define $A(D)$, the area of D, to be

$$A(D) = \sum_{x \in D} m(x)$$

and $L(\partial D)$, the length of the boundary, to be

$$L(\partial D) = \#\{y \sim x \mid x \in D \text{ and } y \notin D\}$$

as in the previous subsection. If we let

$$\alpha = \inf_{\substack{D \subset G \\ D \text{ finite, connected}}} \frac{L(\partial D)}{A(D)}$$

then the main result in Section 2 of [5] states that

Theorem 4.2.1.

$$\lambda_0(\Delta_{bd}) \geq \frac{\alpha^2}{2}.$$

Now, for a general graph G we fix a vertex x_0 of G and let $r(x) = d(x, x_0)$. Then for a vertex x we let

$$\begin{aligned} m_0(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x)\} \\ m_{+1}(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x) + 1\} \\ m_{-1}(x) &= \#\{y \mid y \sim x \text{ and } r(y) = r(x) - 1\} \end{aligned}$$

as before. The main theorem of this section can now be stated as follows:

Theorem 4.2.2. *If for all vertices x of G*

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq c > 0$$

then

$$\lambda_0(\Delta_{bd}) \geq \frac{c^2}{2}.$$

If, in addition, $m(x) \geq m$ then

$$\lambda_0(\Delta) \geq \frac{c^2}{2} m. \tag{4.4}$$

Example 4.2.3. For a tree we have that $m_0(x) = 0$ and $m_{-1}(x) = 1$ for all vertices x so that

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} = \frac{m(x) - 2}{m(x)} = 1 - \frac{2}{m(x)}.$$

Therefore, if $m(x) \geq m > 2$ then $\frac{m_{+1}(x)-m_{-1}(x)}{m(x)} \geq \left(1 - \frac{2}{m}\right)$. Hence, for such a tree, we get $c = \frac{m-2}{m}$, implying

$$\lambda_0(\Delta_{bd}) \geq \frac{(m-2)^2}{2m^2}$$

and

$$\lambda_0(\Delta) \geq \frac{(m-2)^2}{2m}.$$

Proof: Take $D \subset G$ a finite, connected subgraph. Let $r(x) = d(x, x_0)$ where x_0 is a fixed vertex of G. Then by applying the analogue of Green's Theorem we get

$$\begin{aligned} \left|\sum_{x\in D} \Delta_{bd} r(x) m(x)\right| &= \left|\sum_{x\in D} \Delta r(x)\right| \\ &= \left|\sum_{\substack{y\sim x \\ x\in D, y\notin D}} \big(r(x) - r(y)\big)\right| \\ &\leq \sum_{\substack{y\sim x \\ x\in D, y\notin D}} |r(x) - r(y)| \\ &\leq L(\partial D) \end{aligned} \tag{4.5}$$

since $r(x) - r(y)$ can only be ± 1 or 0 if $y \sim x$.

On the other hand,

$$\begin{aligned} \Delta_{bd} r(x) &= r(x) - \frac{1}{m(x)}\Big(m_0(x)r(x) + m_{+1}(x)\big(r(x)+1\big) + m_{-1}(x)\big(r(x)-1\big)\Big) \\ &= \frac{m_{-1}(x) - m_{+1}(x)}{m(x)} \end{aligned}$$

since $m(x) = m_0(x) + m_{+1}(x) + m_{-1}(x)$. Therefore, it follows from the assumption $\frac{m_{+1}(x)-m_{-1}(x)}{m(x)} \geq c$ that $\Delta_{bd} r(x) < 0$ and

$$|\Delta_{bd} r(x)| \geq c$$

for all vertices x. Therefore,

$$\begin{aligned}\left|\sum_{x\in D}\Delta_{bd}r(x)m(x)\right| &= \sum_{x\in D}|\Delta_{bd}r(x)m(x)| \\ &\geq c\sum_{x\in D}m(x) = cA(D). \end{aligned} \tag{4.6}$$

Combining the inequalities (4.5) and (4.6), we get

$$cA(D) \leq L(\partial D)$$

or that

$$c \leq \frac{L(\partial D)}{A(D)}$$

for all finite, connected subgraphs D. Applying Theorem 4.2.1 it follows that

$$\lambda_0(\Delta_{bd}) \geq \frac{c^2}{2}. \tag{4.7}$$

This gives the first part of Theorem 4.2.2.

For the second part of Theorem 4.2.2, we proceed as follows. By using the Rayleigh-Ritz characterization of $\lambda_0(\Delta_{bd})$, inequality (4.7) gives

$$\langle \Delta_{bd}f, f\rangle_{bd} \geq \frac{c^2}{2}\langle f, f\rangle_{bd}$$

for every finitely supported, nonzero function on the graph G. Now,

$$\begin{aligned}\langle \Delta_{bd}f, f\rangle_{bd} &= \sum_{x\in V}\Delta_{bd}f(x)f(x)m(x) \\ &= \sum_{x\in V}\Delta f(x)f(x) = \langle \Delta f, f\rangle\end{aligned}$$

while, if $m(x) \geq m$, then

$$\begin{aligned}\frac{c^2}{2}\langle f, f\rangle_{bd} &= \frac{c^2}{2}\sum_{x\in V}f(x)^2m(x) \\ &\geq \frac{c^2}{2}m\sum_{x\in V}f(x)^2 = \frac{c^2}{2}m\langle f, f\rangle.\end{aligned}$$

Therefore, we get

$$\frac{\langle \Delta f, f\rangle}{\langle f, f\rangle} \geq \frac{c^2}{2}m$$

and taking the infimum over the set of all finitely supported, nonzero functions f it follows that

$$\lambda_0(\Delta) \geq \frac{c^2}{2} m.$$

$\square$

4.3 Essential Spectrum

We now use Theorem 4.2.2 to prove that, under certain assumptions on the graph, $\tilde{\Delta}$, the unique self-adjoint extension of Δ to $\ell^2(V)$, has empty essential spectrum as in [7, Theorem 1.1].

The essential spectrum is, by definition, the complement in the spectrum of the set of isolated eigenvalues of finite multiplicity. We use the notation $\operatorname{spec}(\tilde{\Delta})$ and $\operatorname{ess\,spec}(\tilde{\Delta})$ for the spectrum and essential spectrum of $\tilde{\Delta}$ respectively. Now, as pointed out in [12, Theorem VII.12 and remarks following Theorem VIII.6], the essential spectrum of a self-adjoint operator can be characterized as follows

Theorem 4.3.1. $\lambda \in \operatorname{ess\,spec}(\tilde{\Delta})$ *if and only if there exists an sequence of orthonormal function* $\{f_i\}_{i=0}^{\infty}$ *in the domain of* $\tilde{\Delta}$ *such that*

$$\lim_{i \to \infty} \|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} = 0.$$

In fact, it is sufficient that the sequence be noncompact, that is, have no convergent subsequence, and this will be the characterization of the essential spectrum that we use to prove the theorem below.

Fix a vertex x_0 and let $\underline{m}(r)$ denote the smallest valence of the vertices on the sphere $S_r(x_0)$ as before:

$$\underline{m}(r) = \min_{x \in S_r(x_0)} m(x).$$

We then have

Theorem 4.3.2. *If for all vertices x of G*

$$\frac{m_{+1}(x) - m_{-1}(x)}{m(x)} \geq c > 0$$

and

$$\underline{m}(r) \to \infty \text{ as } r \to \infty$$

then $\tilde{\Delta}$ has empty essential spectrum.

Example 4.3.3. Again, for a tree, we note that if $m(x) \geq m > 2$ for all vertices x then the first assumption is satisfied and so if $\underline{m}(r) \to \infty$ then $\tilde{\Delta}$ will have empty essential spectrum.

The proof of Theorem 4.3.2 will follow easily once we establish the following lemma which is analogous to [7, Proposition 2.1] and apply the second result of Theorem 4.2.2. Let $\tilde{\Delta}_r$ denote the self-adjoint extension of the Laplacian acting on the space $C_0(V, B_r)$, of functions with finite support disjoint from B_r, to $\ell^2(V, B_r)$, the square summable functions which vanish on B_r. We then have that

Lemma 4.3.1. *$\tilde{\Delta}$ and $\tilde{\Delta}_r$ have the same essential spectrum.*

Assuming the lemma for now, we give the proof of the theorem:

Proof of Theorem 4.3.2: By applying (4.4) from Theorem 4.2.2 we get that

$$\lambda_0(\tilde{\Delta}_r) \to \infty \text{ as } r \to \infty$$

since $\underline{m}(r) \to \infty$. Now, applying Lemma 4.3.1, since the essential spectrum of $\tilde{\Delta}$ is the same as that of $\tilde{\Delta}_r$ and the bottom of the spectrum of $\tilde{\Delta}_r$ is increasing to infinity, it must follow that the essential spectrum of $\tilde{\Delta}$ is empty. $\square$

Proof of Lemma 4.3.1: Let $\lambda \in$ ess spec $(\tilde{\Delta}_r)$. Let $\{f_i\}_{i=0}^{\infty}$ be a sequence of orthonormal functions vanishing on B_r satisfying

$$\lim_{i\to\infty} \|\tilde{\Delta}_r f_i - \lambda f_i\|_{\ell^2} = 0.$$

Then, since

$$\tilde{\Delta}_r f_i(x) \neq \tilde{\Delta} f_i(x) \text{ only for } x \in \partial B_r$$

and, by orthonormality, for every vertex x, $f_i(x) \to 0$ as $i \to \infty$ it follows that

$$\lim_{i\to\infty} \|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} = 0$$

so that $\lambda \in \text{ess spec }(\tilde{\Delta})$.

Now, say that $\lambda \in \text{ess spec}(\tilde{\Delta})$ and $\{f_i\}_{i=0}^{\infty}$ is a sequence of orthonormal function such that

$$\|\tilde{\Delta} f_i - \lambda f_i\|_{\ell^2} \to 0 \text{ as } i \to \infty.$$

Let

$$\varphi_r(x) = \begin{cases} 0 & \text{if } x \in B_r(x_0) \\ 1 & \text{otherwise} \end{cases}$$

We claim that $\{\varphi_r f_i\}_{i=0}^{\infty}$ will be a sequence of bounded functions with no convergent subsequence satisfying

$$\|\tilde{\Delta}_r(\varphi_r f_i) - \lambda(\varphi_r f_i)\|_{\ell_2} \to 0 \text{ as } i \to \infty.$$

To show that $\{\varphi_r f_i\}_{i=0}^{\infty}$ has no convergent subsequences we first note that since $\{f_i\}_{i=0}^{\infty}$ are orthonormal, $\{f_i\}_{i=0}$ has no convergent subsequences. This follows since pointwise, using orthonormality as above,

$$f_i(x) \to 0 \text{ as } i \to \infty \quad \text{for all } x \in V$$

while $\|f_i\|_{\ell^2} = 1$ for all i. Now, assume that $\{\varphi_r f_i\}_{i=0}^{\infty}$ has a convergent subsequence, say

$$\varphi_r f_{i_k} \to f \text{ as } k \to \infty \text{ in } \ell_2.$$

Since $f_i \in \ell_2$, $\{f_{i_k}(x)\}_{k=0}^{\infty}$ has a convergent subsequence for each x. Because B_r has only finitely many vertices, we can find a subsequence of $\{f_{i_k}\}_{k=0}^{\infty}$ which converges for each $x \in B_r$. We continue to denote this subsequence as $\{f_{i_k}\}_{k=0}^{\infty}$ and let $\hat{f}(x)$ be defined by

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } x \notin B_r \\ \lim_{k\to\infty} f_{i_k}(x) & \text{if } x \in B_r \end{cases}$$

Then, it would follow that

$$\begin{aligned}
\|f_{i_k} - \hat{f}\|_{\ell_2}^2 &= \sum_{x\in V} \big(f_{i_k}(x) - \hat{f}(x)\big)^2 \\
&= \sum_{x\notin B_r} \big(f_{i_k}(x) - f(x)\big)^2 + \sum_{x\in B_r} \big(f_{i_k}(x) - \hat{f}(x)\big)^2 \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

so that $\{f_i\}_{i=0}^{\infty}$ would have a convergent subsequence. The contradiction shows that $\{\varphi_r f_i\}_{i=0}^{\infty}$ cannot have a convergent subsequence.

What remains to be shown is that $\|(\tilde{\Delta}_r - \lambda I)(\varphi_r f_i)\|_{\ell^2} \to 0$ as $i \to \infty$. First, we calculate $\tilde{\Delta}_r(\varphi_r f_i)$:

$$\begin{aligned}
\tilde{\Delta}_r(\varphi_r f_i)(x) &= \sum_{y\sim x} \big(\varphi_r(x) f_i(x) - \varphi_r(y) f_i(y)\big) \\
&= \sum_{y\sim x} \big(\varphi_r(x) f_i(x) - \varphi_r(x) f_i(y) + \varphi_r(x) f_i(y) - \varphi_r(y) f_i(y)\big) \\
&= \varphi_r(x)\tilde{\Delta}_r f_i(x) + \sum_{y\sim x} f_i(y)\big(\varphi_r(x) - \varphi_r(y)\big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{x\in V} \big(\tilde{\Delta}_r(\varphi_r f_i)(x) - \lambda(\varphi_r f_i)(x)\big)^2 &\leq \sum_{x\in V} \Big(\varphi_r(x)\big((\tilde{\Delta}_r f_i)(x) - \lambda f_i(x)\big)\Big)^2 \\
&\quad + \sum_{x\in V}\sum_{y\sim x} \Big(f_i(y)\big(\varphi_r(x) - \varphi_r(y)\big)\Big)^2 \\
&= \sum_{x\notin B_r} \big((\tilde{\Delta} f_i)(x) - \lambda f_i(x)\big)^2 \\
&\quad + \sum_{x\in \partial B_r} \sum_{\substack{y\sim x \\ y\notin B_r}} \big(f_i(x)^2 + f_i(y)^2\big).
\end{aligned}$$

Now, the first sum above goes to 0 as $i \to \infty$ by the assumption on f_i while, since f_i are orthonormal, $f_i(x) \to 0$ as $i \to \infty$ for each x, so it follows that the second sum also goes to 0. Therefore,

$$\|\tilde{\Delta}_r(\varphi_r f_i) - \lambda(\varphi_r f_i)\|_{\ell_2} \to 0 \text{ as } i \to \infty$$

and so $\lambda \in \text{ess spec}(\tilde{\Delta}_r)$. $\square$

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